Critical Random graphs and Applications

Louigi Addario-Berry, *McGill University*

Nicolas Broutin, *Inria Paris-Rocquencourt*

Christina Goldschmidt, *University of Oxford*

Grégory Miermont, *ENS Lyon*

Journées Boole, June 2013

Random graphs, phase transition

The structure of critical random graphs

The minimum spanning tree

More optimization problems

Erdős–Rényi random graphs

Definition. Random graph $G(n, p)$

graph on $\{1, 2, \ldots, n\}$

every edge is present with probability p

C_i^n the connected components in decreasing order of size

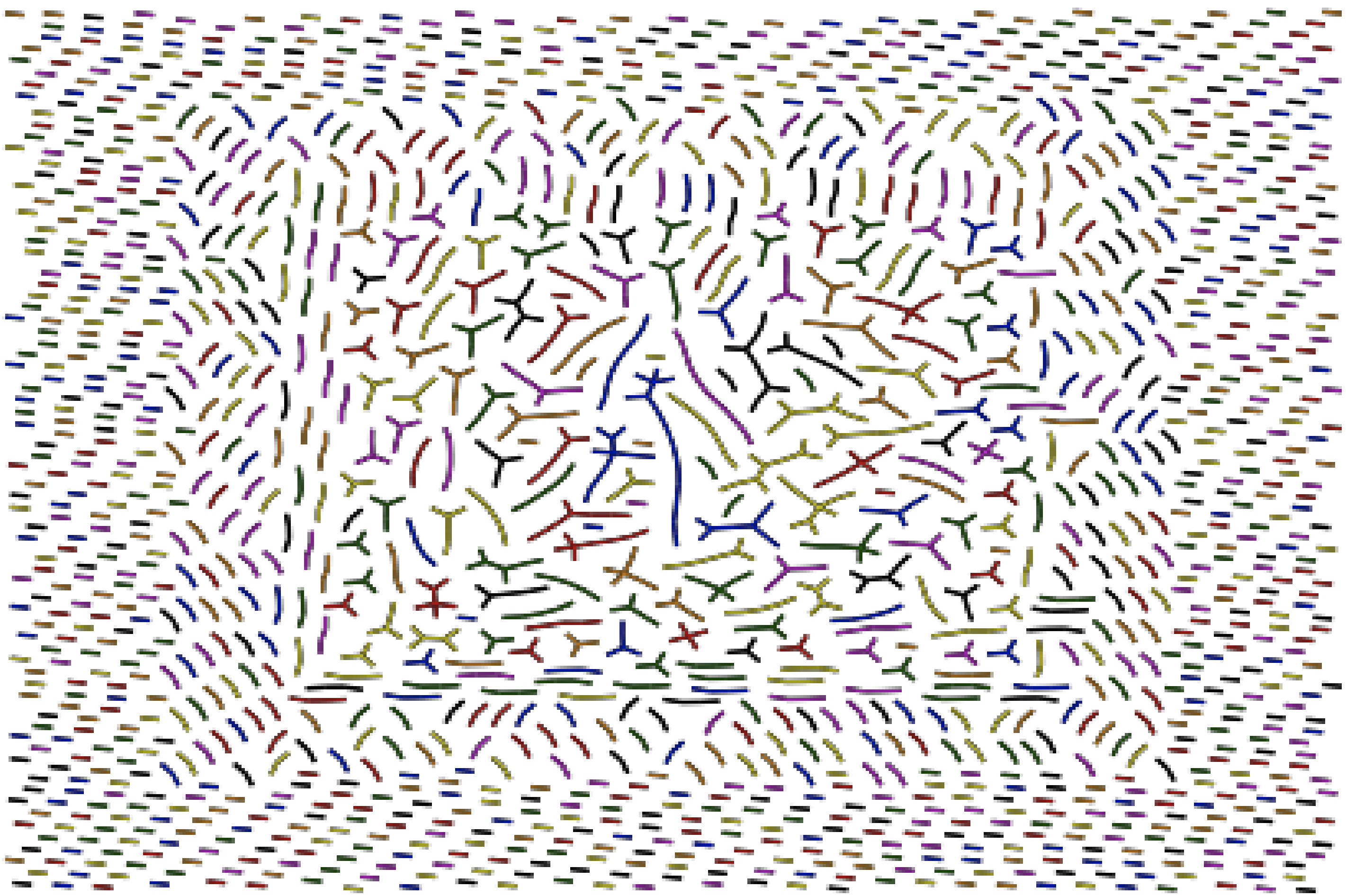
Phase transition: $G(n, c/n)$

$c < 1$: $|C_1^n| = O(\log n)$

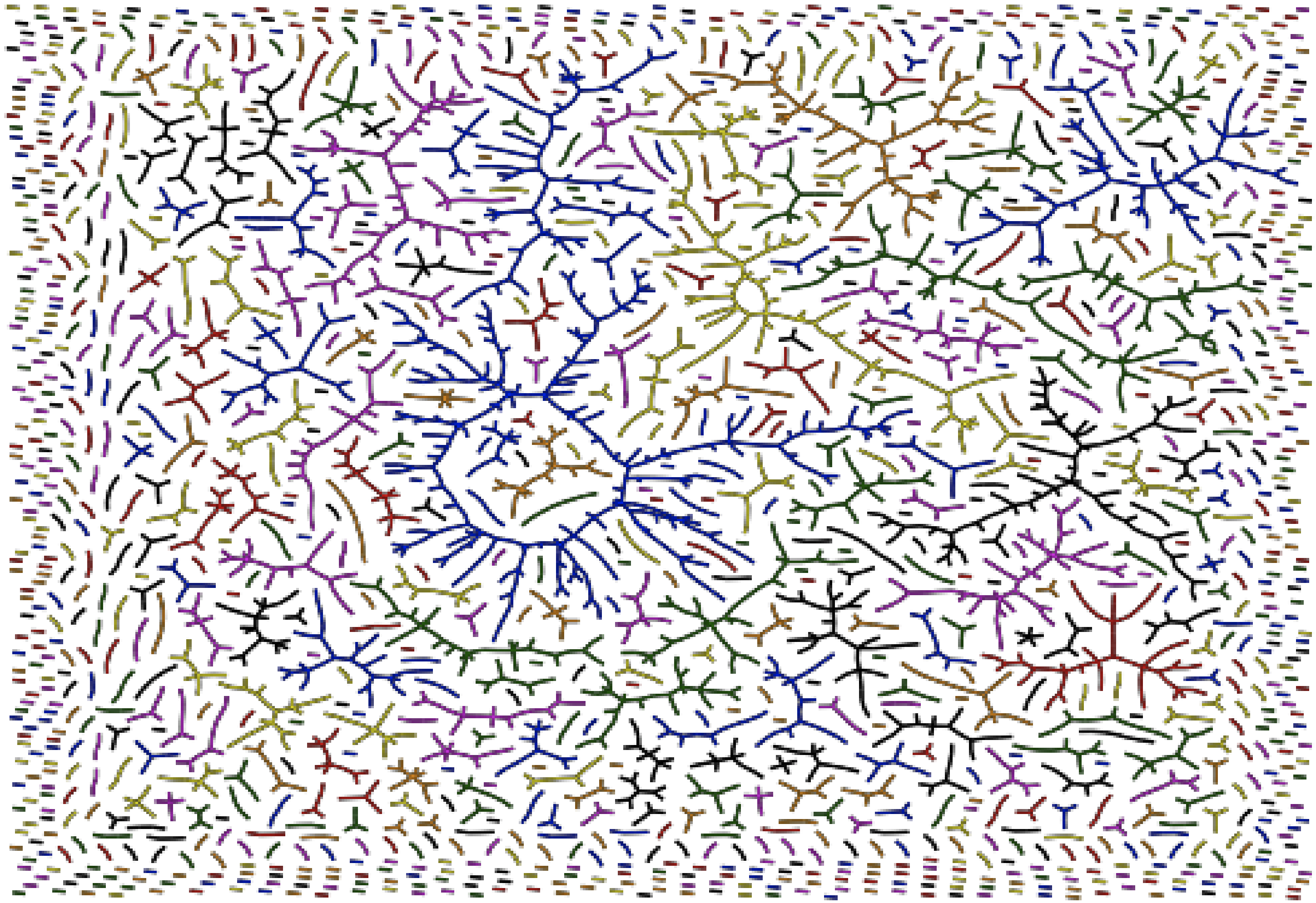
$c = 1$: $|C_1^n|, |C_2^n|, \ldots, |C_k^n| \approx n^{2/3}$

$c > 1$: $|C_1^n| = \Omega(n), \quad |C_2^n| = O(\log n)$

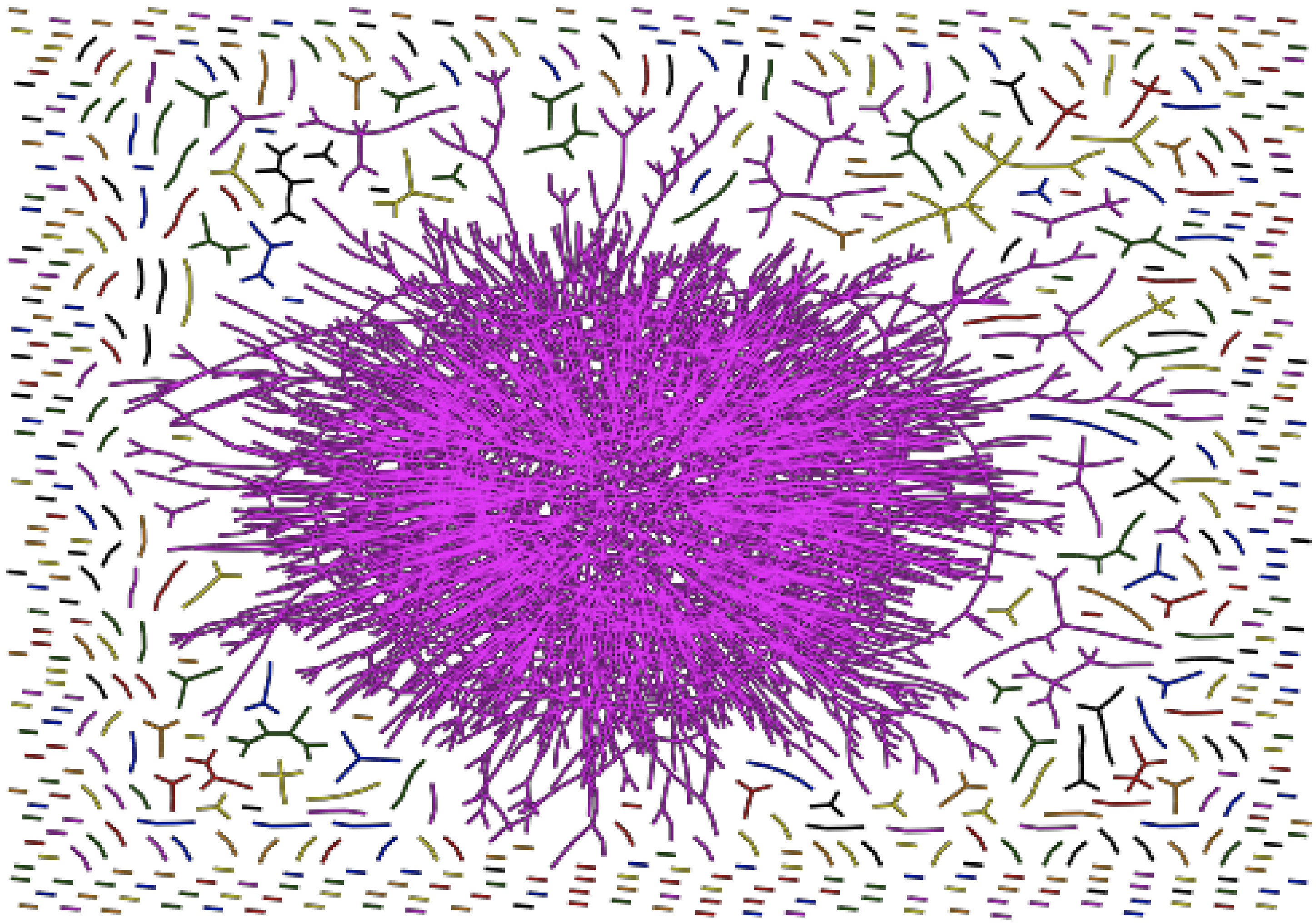
$G(10000, \frac{0.5}{10000})$



$$G(10000, \frac{1.0}{10000})$$



$G(10000, \frac{1.5}{10000})$



The phase transition

inside the critical window $pn = 1 + \lambda n^{-1/3}$

Theorem. (Aldous 1997)

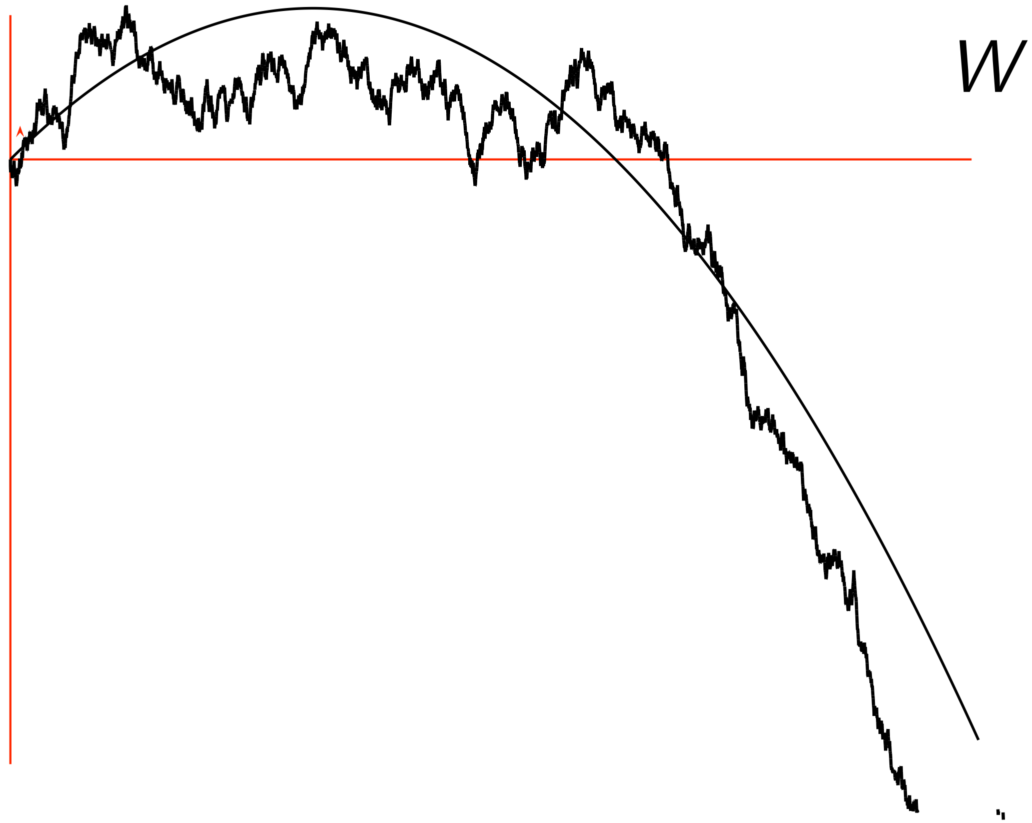
$$\left(n^{-2/3}|C_i^n|, s(C_i^n)\right)_{i\geq 1} \to \left(|\gamma_i|, s(\gamma_i)\right)_{i\geq 1}$$

The phase transition

inside the critical window $pn = 1 + \lambda n^{-1/3}$

Theorem. (Aldous 1997)

$$\left(n^{-2/3}|C_i^n|, s(C_i^n)\right)_{i\geq 1} \to \left(|\gamma_i|, s(\gamma_i)\right)_{i\geq 1}$$



$$W^\lambda(t) := t\lambda - t^2/2 + W(t)$$

W standard BM

The phase transition

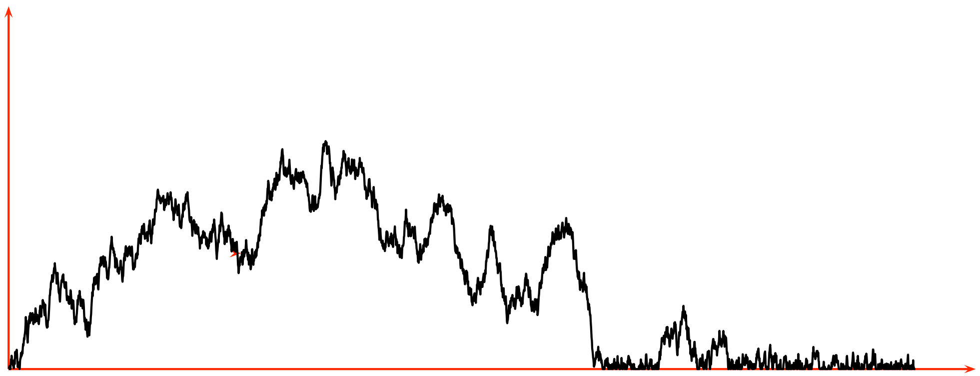
inside the critical window $pn = 1 + \lambda n^{-1/3}$

Theorem. (Aldous 1997)

$$(n^{-2/3}|C_i^n|, s(C_i^n))_{i\geq 1} \to (|\gamma_i|, s(\gamma_i))_{i\geq 1}$$

$$W^\lambda(t) := t\lambda - t^2/2 + W(t)$$

W standard BM



$$B^\lambda(t) := W^\lambda(t) - \inf_{0\leq s\leq t} W^\lambda(s)$$

The phase transition

inside the critical window $pn = 1 + \lambda n^{-1/3}$

Theorem. (Aldous 1997)

$$(n^{-2/3}|C_i^n|, s(C_i^n))_{i\geq 1} \to (|\gamma_i|, s(\gamma_i))_{i\geq 1}$$

$$W^\lambda(t) := t\lambda - t^2/2 + W(t)$$

W standard BM

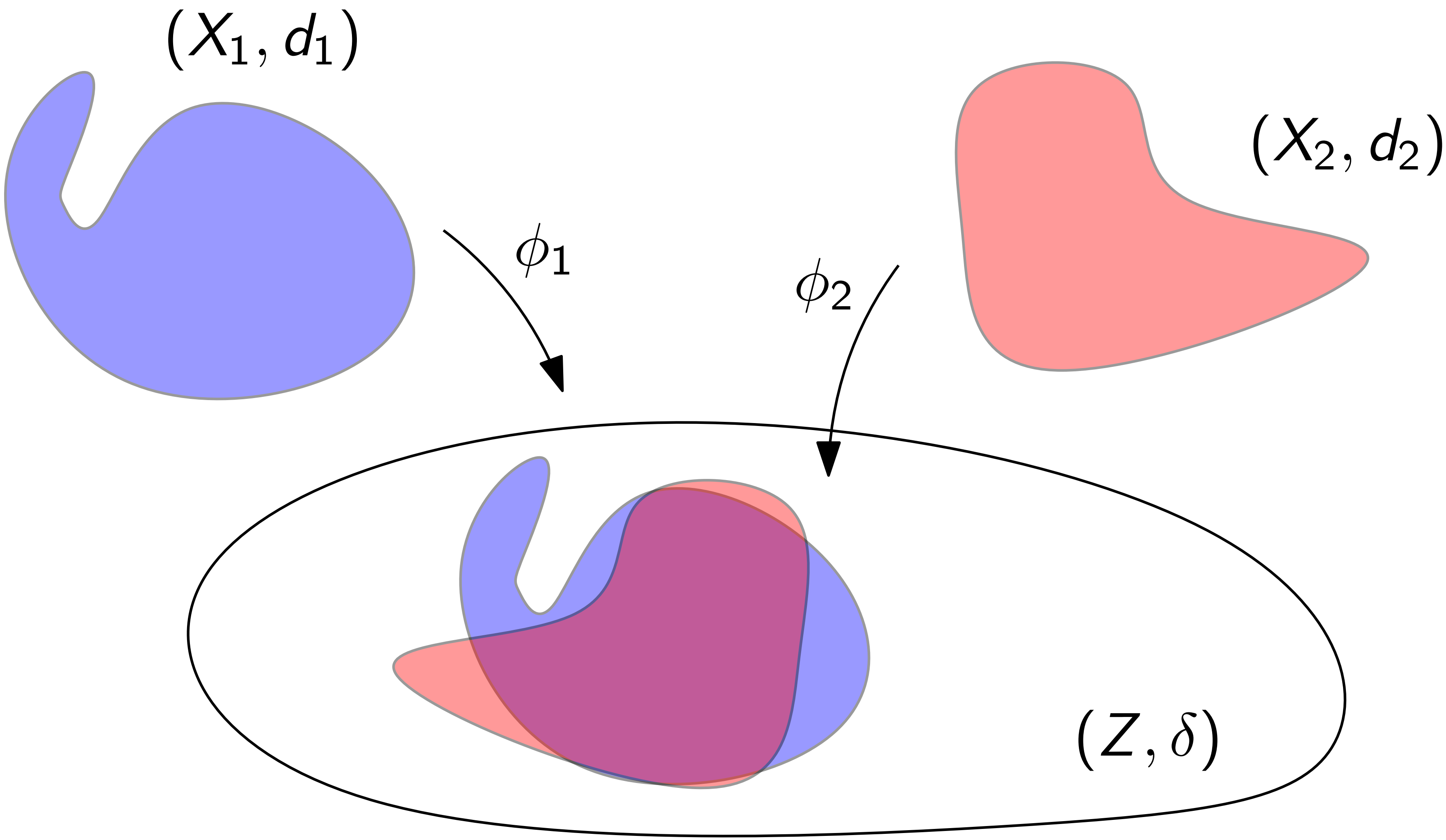
$$B^\lambda(t) := W^\lambda(t) - \inf_{0\leq s\leq t} W^\lambda(s)$$

Poisson point process rate one in $[0,\infty) \times [0,\infty)$

Phase transitions and fractals

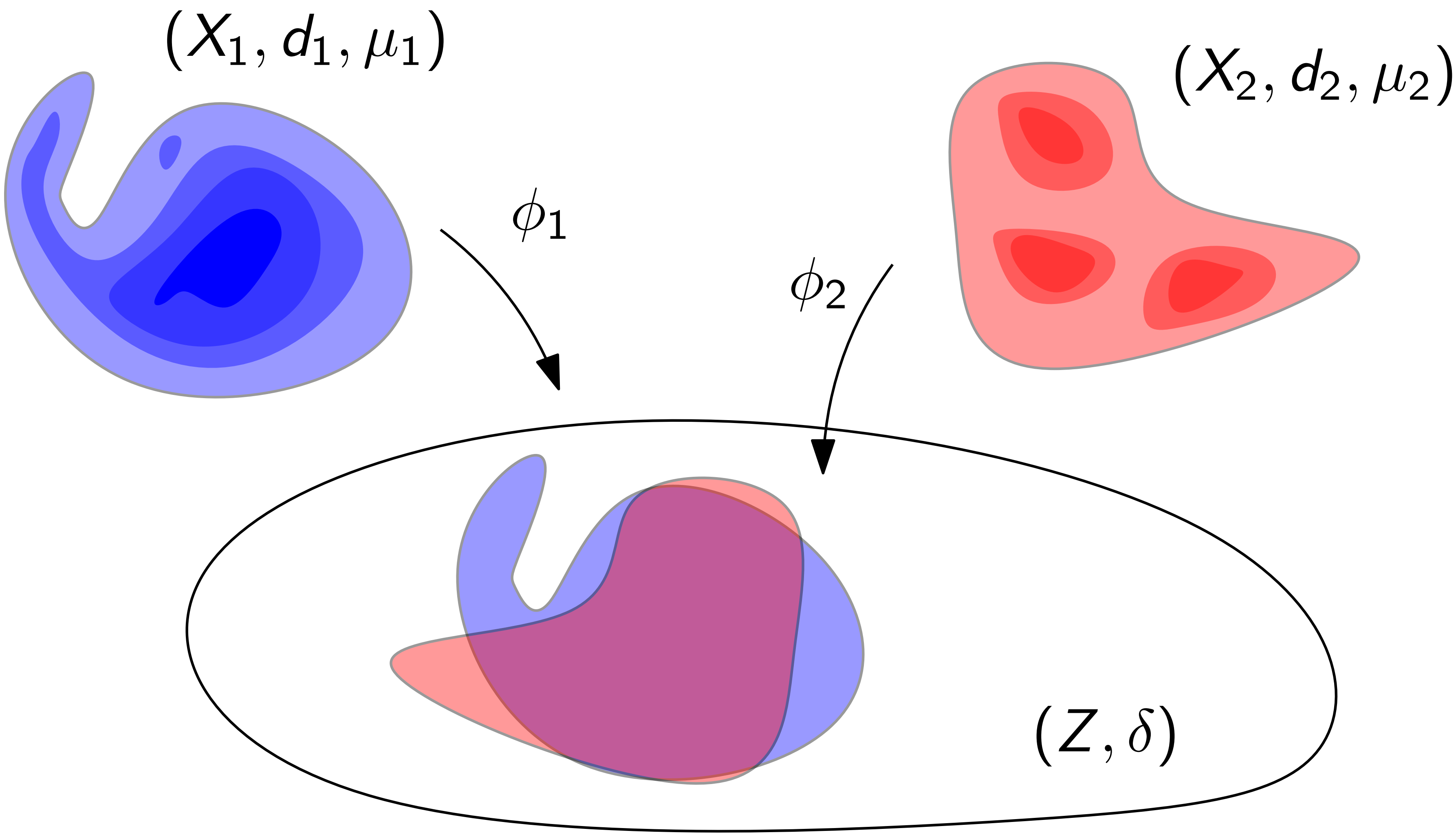
Comparing metric spaces

Gromov-Hausdorff topology.



Comparing measured metric spaces

Gromov-Hausdorff-Prokhorov topology.



Scaling critical random graphs

critical window $G(n,p)$, for $pn = 1 + \lambda n^{-1/3}$

C_i^n the ith largest c.c.

distances rescaled by $n^{-1/3}$

mass $n^{-2/3}$ on each vertex

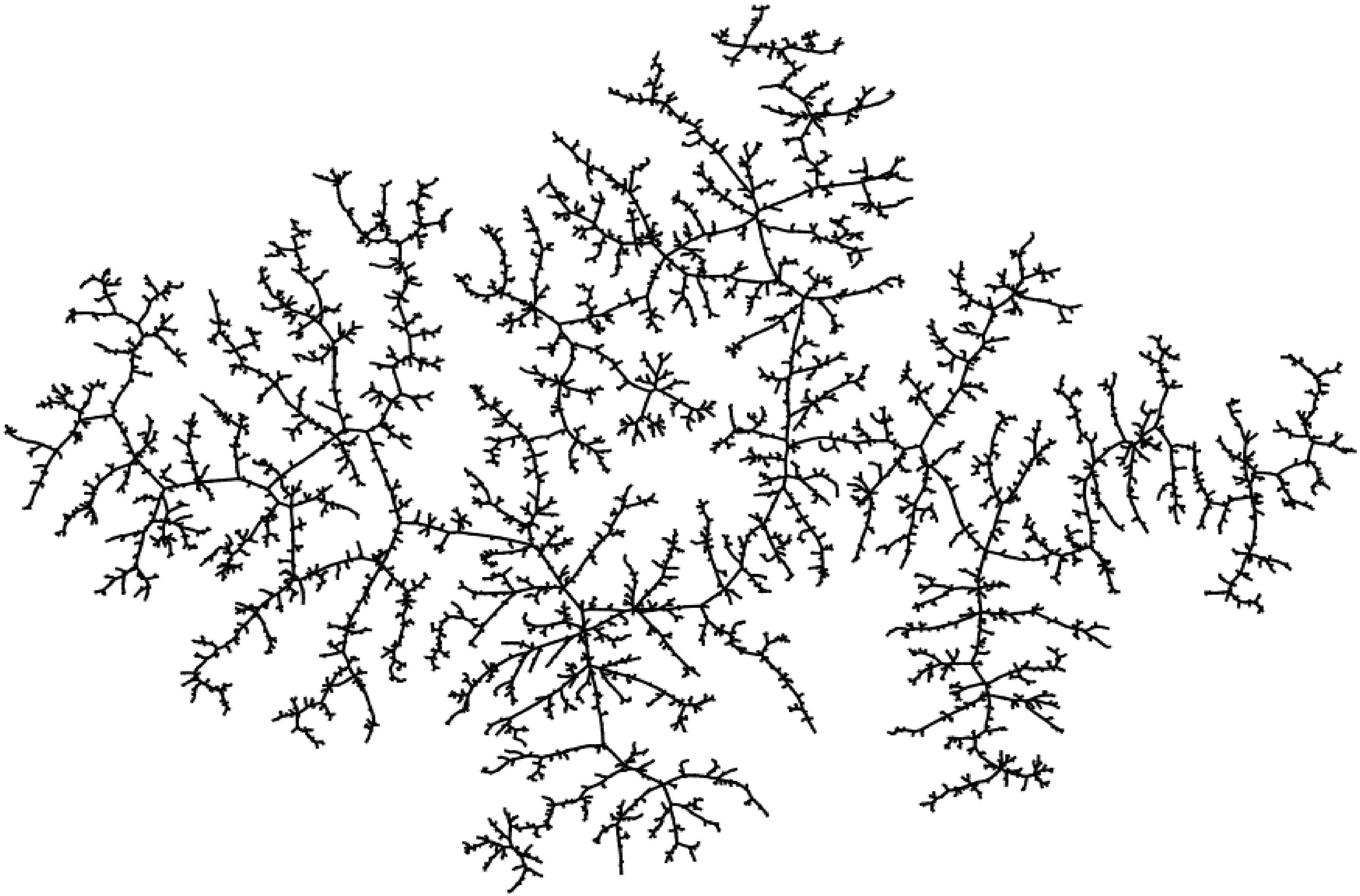
Theorem. (ABG2012)

$$(C_i^n)_{i\geq 1} \to (\mathscr{C}_i)_{i\geq 1}$$

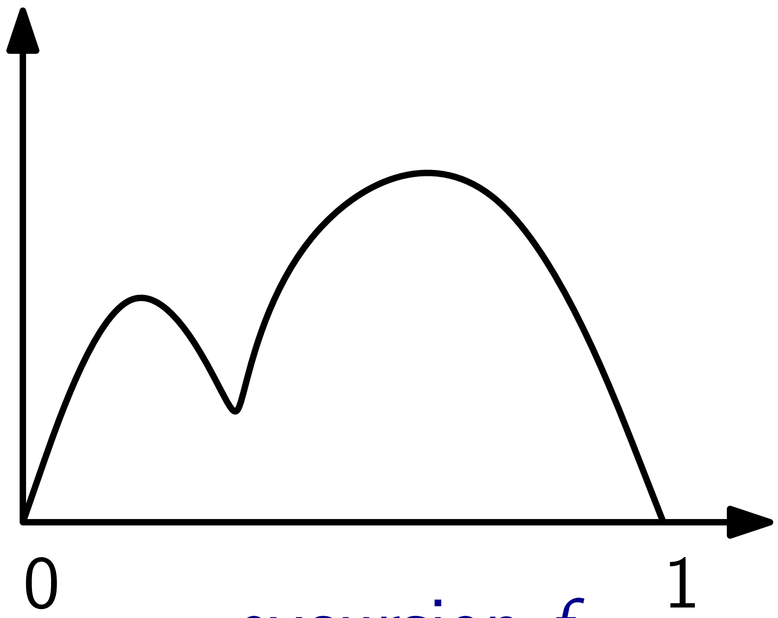
in distribution for the "GHP distance"

$$d_{GHP}^4(\mathbf{A}, \mathbf{B}) = \left(\sum_{i\geq 1} \mathrm{d}_{\mathrm{GHP}}(A_i, B_i)^4 \right)^{1/4}$$

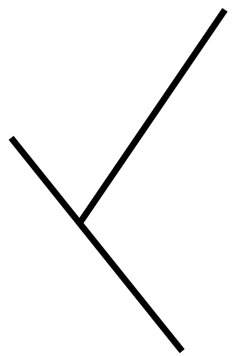
A (limit) random connected component



The tree encoded by a Brownian excursion (CRT)



excursion f



tree $\mathcal{T}_f$

Definition: $(\mathcal{T}_f, d_f)$

$$d_f(x,y) = f(x) + f(y) - 2\inf\{f(t) : x \wedge y \leq t \leq x \vee y\}$$

$x \sim_f y$ if $d_f(x,y) = 0$ $\qquad$ $([0,1]/\sim_f, d_f)$ is a real tree

Scaling limit of random trees

Theorem. (Aldous)

T_n a uniformly random tree on $\{1, 2, \ldots, n\}$

$n^{-1/2} T_n \to \mathcal{T}_{2e}$

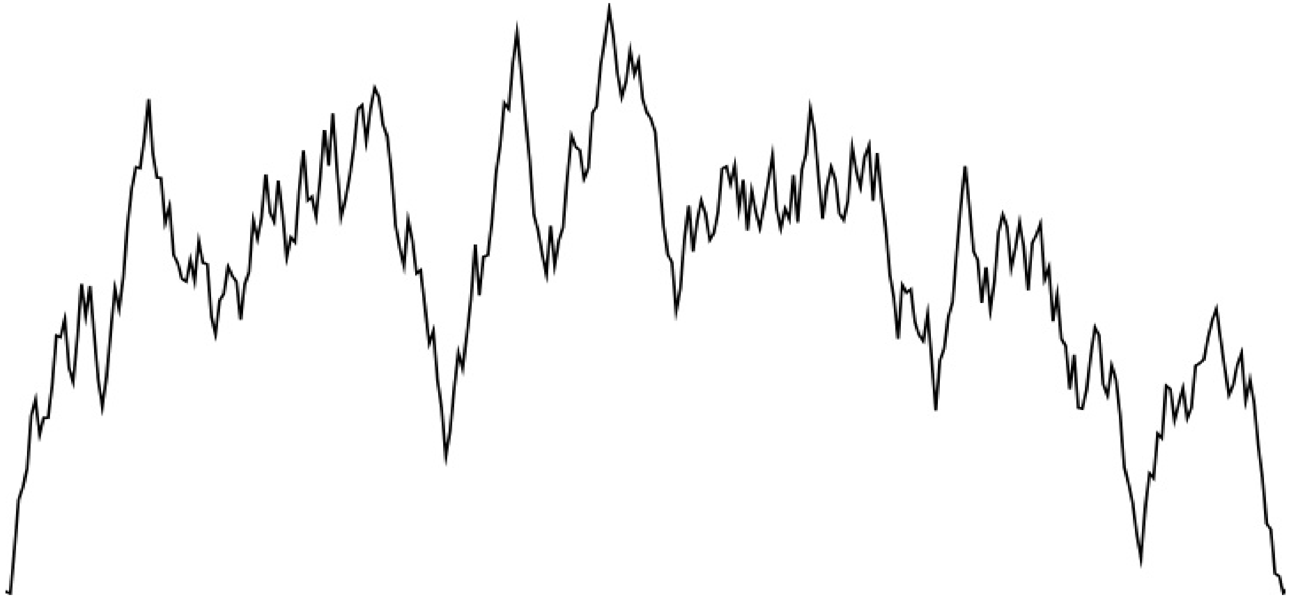
Scaling limit of random trees

Theorem. (Aldous)

T_n a uniformly random tree on $\{1, 2, \ldots, n\}$

$n^{-1/2} T_n \to \mathcal{T}_{2e}$

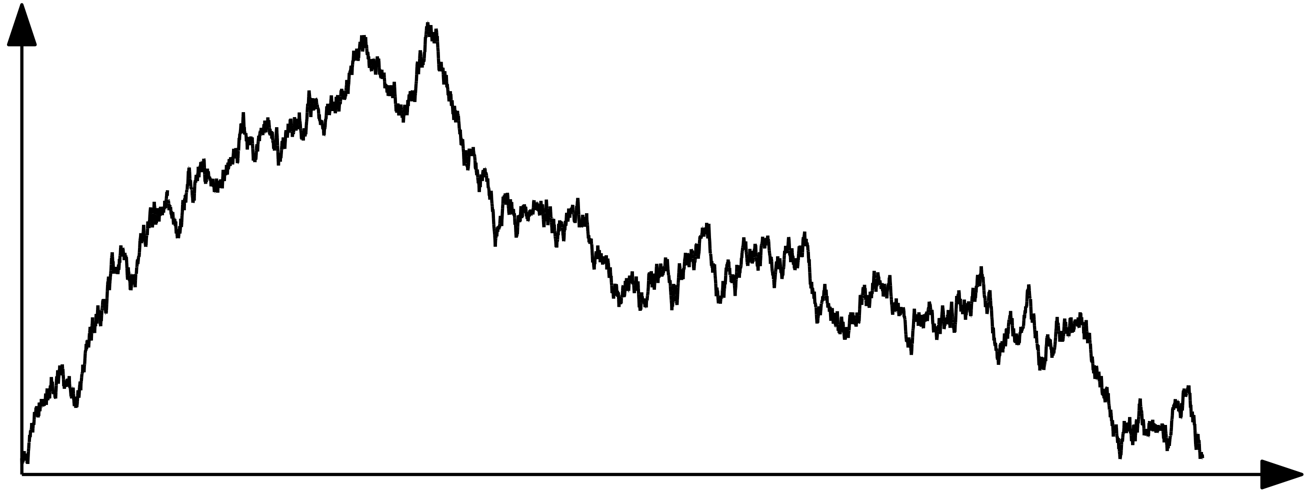
$\mathcal{T}_{2\mathbf{e}}$: **Continuum random tree**

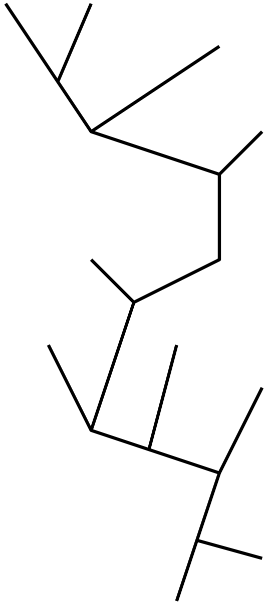


A limit connected component I

Tilted excursions

$$\mathbf{E}[f(\tilde{e})] = \frac{\mathbf{E}[f(e)\exp(\int_0^1 e(s)ds)]}{\mathbf{E}[\exp(\int_0^1 e(s)ds)]}$$

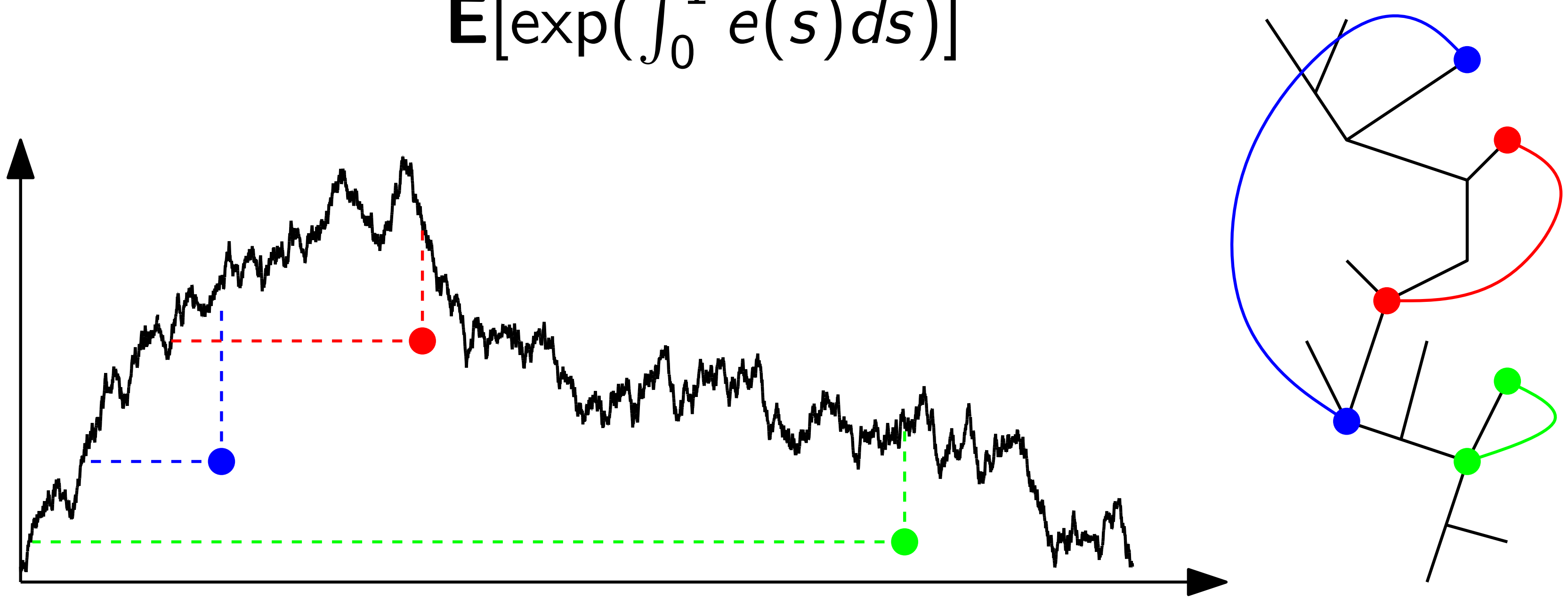




A limit connected component I

Tilted excursions

$$\mathbf{E}[f(\tilde{e})] = \frac{\mathbf{E}[f(e)\exp(\int_0^1 e(s)ds)]}{\mathbf{E}[\exp(\int_0^1 e(s)ds)]}$$



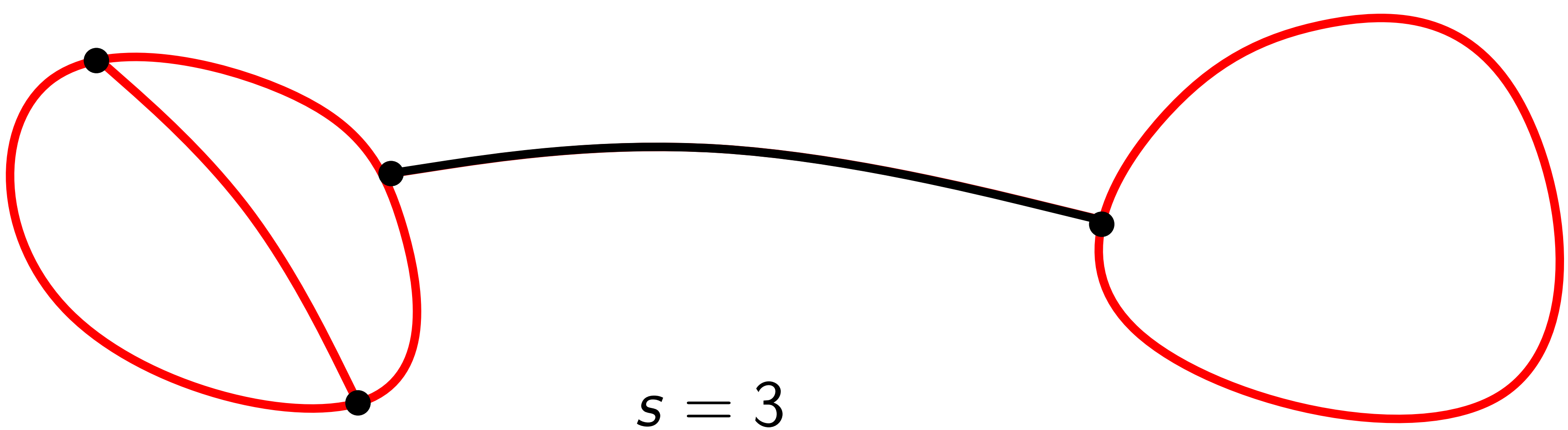
Poisson process rate one under $\tilde{e}$

For each point $\{\color{blue}{\bullet}, \color{green}{\bullet}, \color{red}{\bullet}\}$ *identify* two point of $\mathcal{T}_{2\tilde{e}}$

A limit connected component II

1. Sample a connected 3 regular multigraph with $2(s-1)$ vertices and $3(s-1)$ edges
2. respective masses of the bits:

   Sample a vector $(X_1, \ldots, X_{3(s-1)}) \sim \text{Dirichlet}(\frac{1}{2}, \ldots, \frac{1}{2})$
3. sample $3(s-1)$ independent CRT with 2 distinguished points each

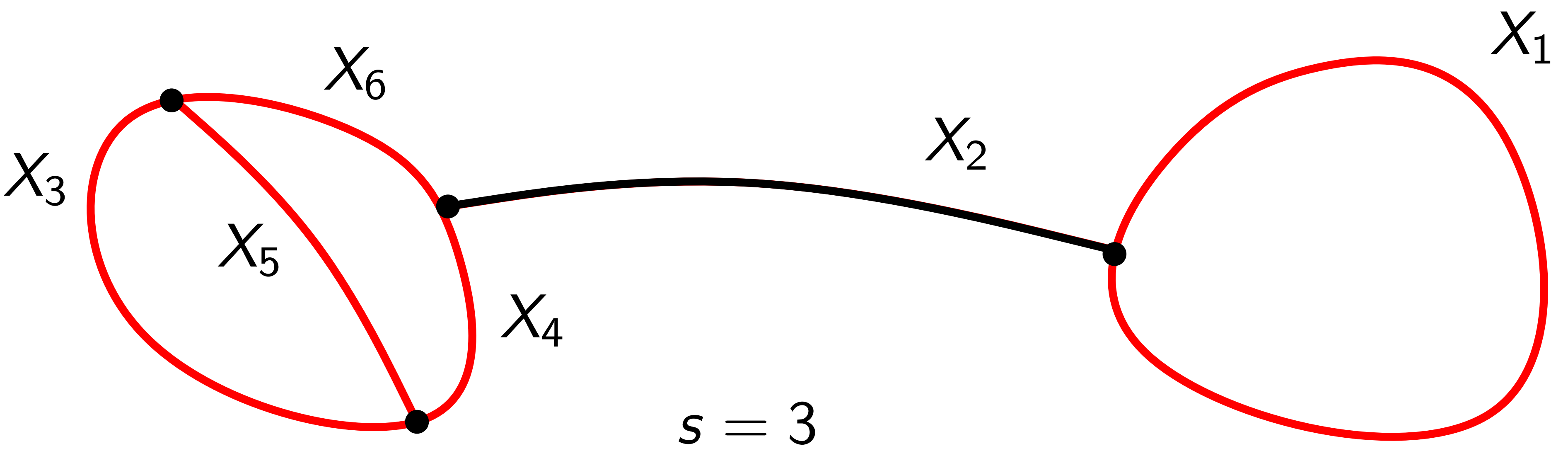


$s = 3$

A limit connected component II

1. Sample a connected 3 regular multigraph with $2(s-1)$ vertices and $3(s-1)$ edges
2. respective masses of the bits:

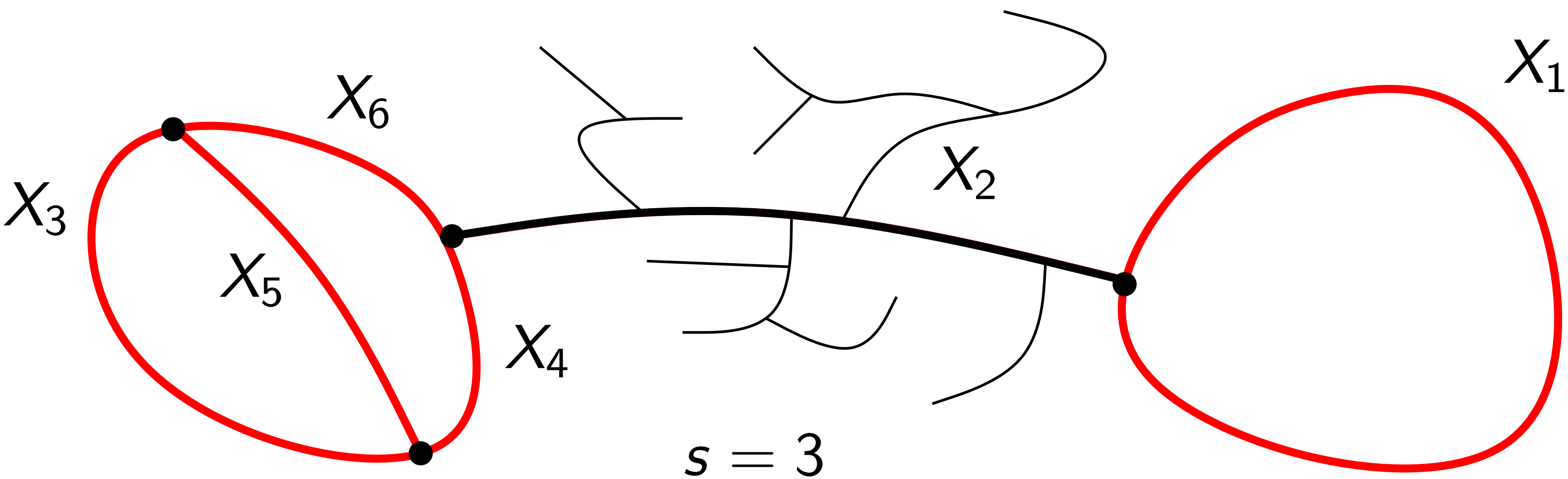
   Sample a vector $(X_1, \ldots, X_{3(s-1)}) \sim \text{Dirichlet}(\frac{1}{2}, \ldots, \frac{1}{2})$
3. sample $3(s-1)$ independent CRT with 2 distinguished points each



A limit connected component II

1. Sample a connected 3 regular multigraph with $2(s-1)$ vertices and $3(s-1)$ edges
2. respective masses of the bits:

   Sample a vector $(X_1, \ldots, X_{3(s-1)}) \sim \text{Dirichlet}(\frac{1}{2}, \ldots, \frac{1}{2})$
3. sample $3(s-1)$ independent CRT with 2 distinguished points each



The minimum spanning tree

Definition.

$G = (V, E)$ a connected graph

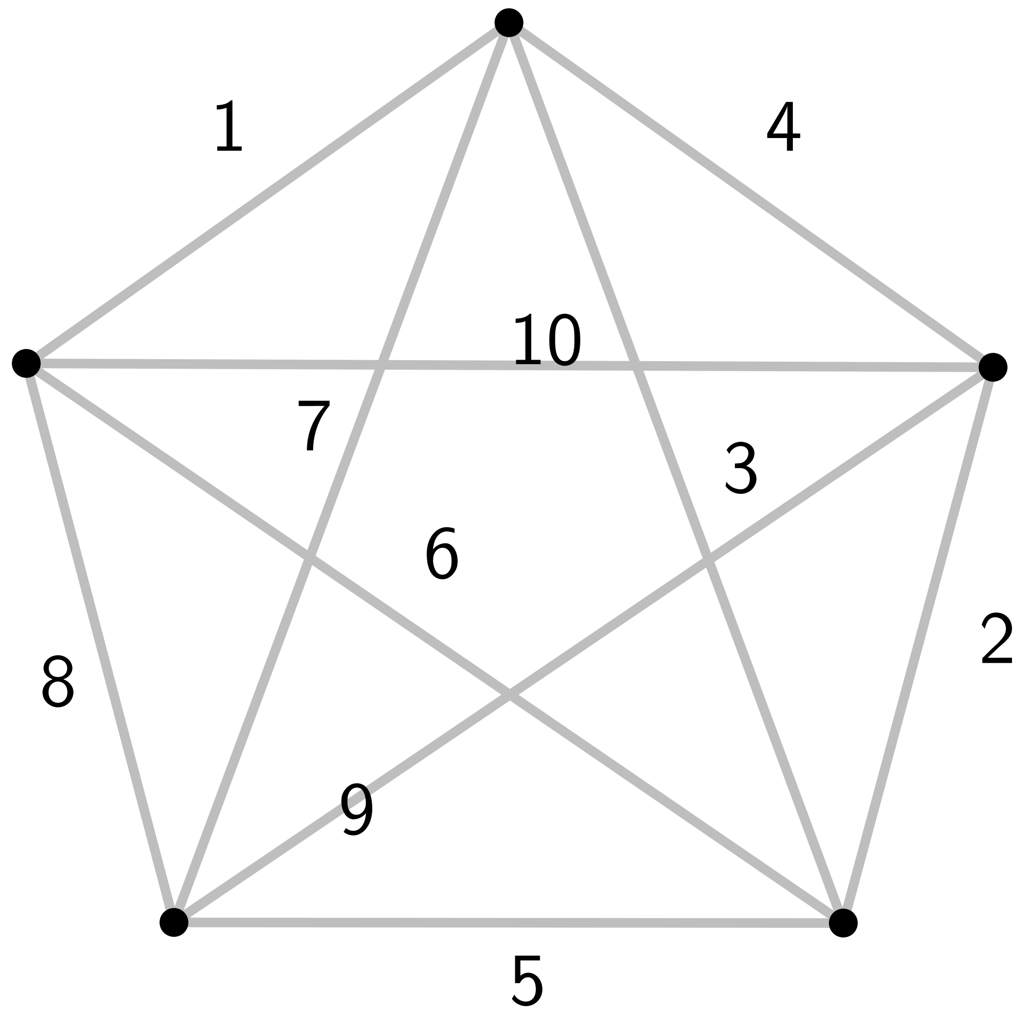
$w_e \geq 0, e \in E$ weights

MST $=$ lightest connected subgraph of G

Kruskal's algorithm.

1. sort the edges by increasing weight, e_i, $1 \leq i \leq |E|$
2. Initially set $T_0 = (V, \emptyset)$
3. Set $T_{i+1} = T_i \cup \{e_i\}$ iff it does not create a cycle

Kruskal – Example



Random Model

"Mean-field" model

graph: complete graph K_n

iid uniform weights

A little history.

Frieze ('85): total weight converges to $\zeta(3)$

Janson ('95): CLT

Aldous: degree of the node 1

Random Model

"Mean-field" model

graph: complete graph K_n

iid uniform weights

A little history.

Frieze ('85): total weight converges to $\zeta(3)$

Janson ('95): CLT

Aldous: degree of the node 1

But... all these informations are local

What is the global *metric* structure?

The continuum spanning tree

The rescaled minimum spanning tree

T_n the minimum spanning tree of K_n

$n^{-1/3}d_n$, for d_n the graph distance

μ_n mass $1/n$ on each vertex of $\{1, 2, \ldots, n\}$

Theorem (ABGM 2013)

There exists a random compact metric space $\mathscr{M}$

such that:
$$T_n \xrightarrow[GHP]{d} \mathscr{M}$$

A few properties of $\mathscr{M}$

Proposition.

1. $\mathscr{M}$ is geodesic
2. $\mathscr{M}$ has no loop
3. $\mathscr{M}$ has maximum degree 3
4. for μ-almost every x, $\deg(x) = 1$

A few properties of $\mathscr{M}$

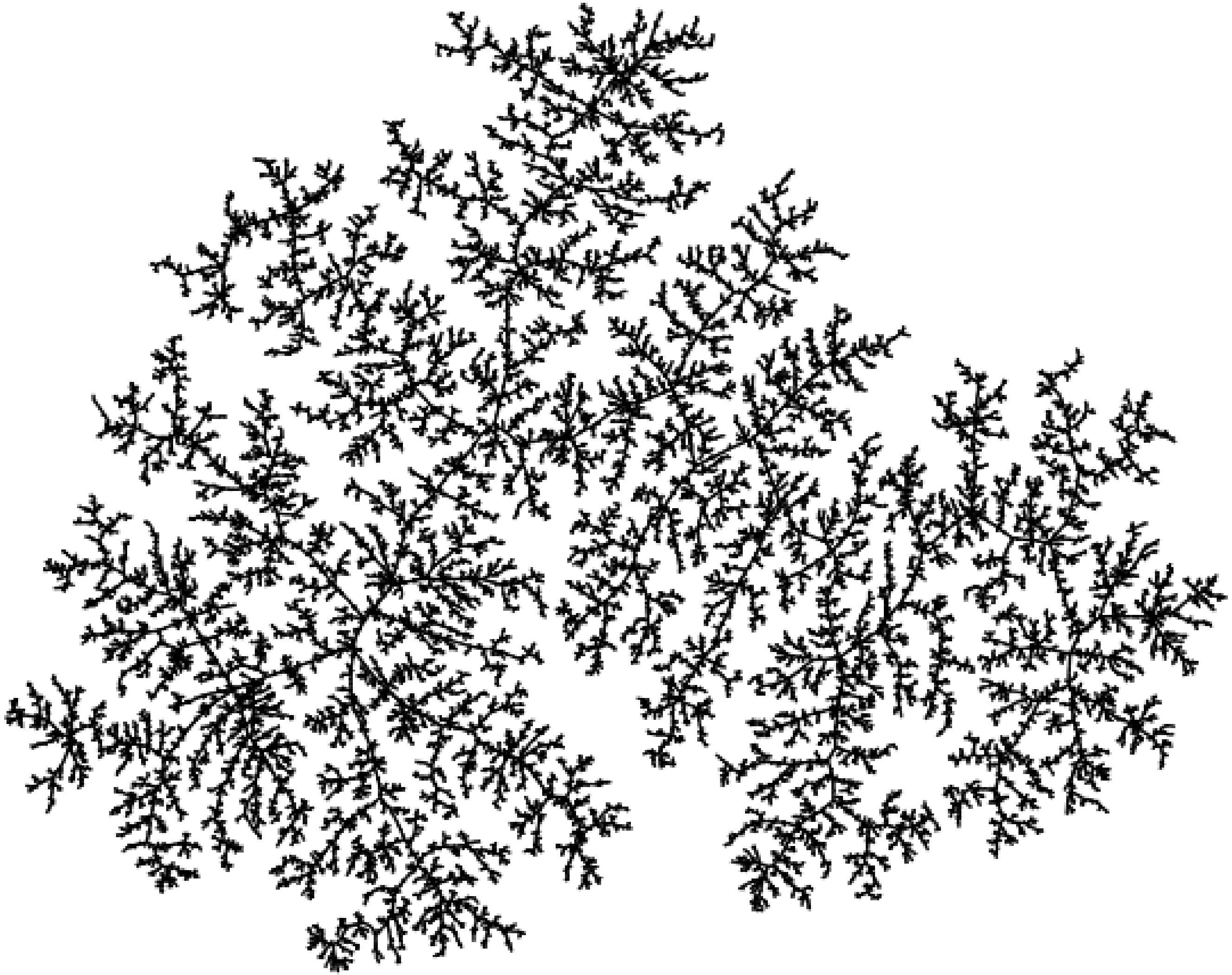
Proposition.

1. $\mathscr{M}$ is geodesic
2. $\mathscr{M}$ has no loop
3. $\mathscr{M}$ has maximum degree 3
4. for μ-almost every x, $\deg(x) = 1$

Proposition.

$\mathscr{M}$ is not Aldous' Continuum Random Tree

What does it look like?



(X, d) a compact metric space

$N(X, r) =$ min number of balls of radius r to cover X

$$\underline{\dim}(X) = \liminf_{r \to 0} \frac{\log N(X, r)}{\log(1/r)} \qquad \overline{\dim}(X) = \limsup_{r \to 0} \frac{\log N(X, r)}{\log(1/r)}$$

box-counting dimension

$\dim(X)$ is the common value, if they are equal

Example:

$N([0,1], r) \approx 1/r$ $\qquad$ $\dim([0,1]) = 1$

Dimensions of continuum random trees

Theorem.

$$\dim(\mathscr{M}) = 3 \qquad \text{with probability one}$$

while

Theorem.

$$\dim(CRT) = 2 \text{ with probability one}$$

Forward-Backward approach

Two main tools. In Kruskal's algorithm

Track the metric structure as the edges are added.

Track the metric structure as the edges are removed

Strategy.

1. Build $G(n, p)$: Add all edges until some weight p
2. Remove the edges that should not have been put

Forward-Backward approach

Two main tools. In Kruskal's algorithm

Track the metric structure as the edges are added.

Track the metric structure as the edges are removed

Strategy.

1. Build $G(n, p)$: Add all edges until some weight p
2. Remove the edg[illegible]ld not have been put

2'. Conditional on $G(n, p) = G$,

construct a tree **distributed** as MST(G)

When is the metric structure built?

$T(n,p)$ portion of the MST that is in $G(n,p)$

(Here d_{GHP} compares two *sequences* of cc)

Evolution of distances:

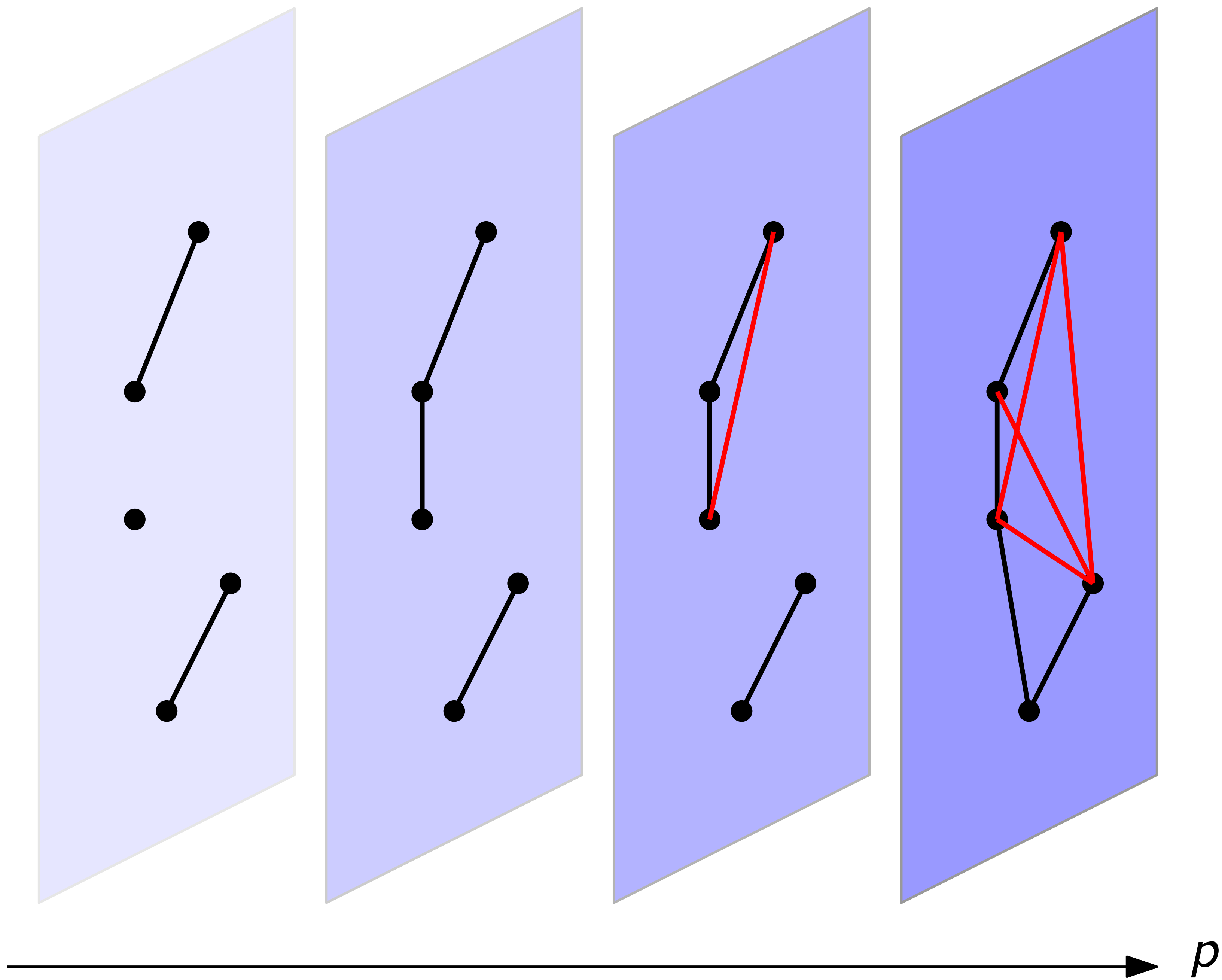
for all $p < (1-\epsilon)/n$

$$\mathrm{d}_{\mathrm{GHP}}(T(n,p);\underline{0}) = O(\log n)$$

for all $p > (1+\epsilon)/n$

$$\mathrm{d}_{\mathrm{GHP}}(T(n,p);((T(n,1),\underline{0})) = O(\log^{10} n)$$

Look around the critical phase


p

Some other optimization problems

2-XOR-SAT

n boolean variables

Each constraint $x_i \oplus x_j = *$ present with proba p

SAT iif no cycle of odd weight

$$\mathbf{P}(SAT) = \mathbf{E}\left[2^{-\#\{\text{Poisson points}\}}\right] \cdot \mathbf{E}\left[2^{-\#\{\text{small unicyclic}\}}\right]$$

Some other optimization problems

2-XOR-SAT

n boolean variables

Each constraint $x_i \oplus x_j = *$ present with proba p

SAT iif no cycle of odd weight

$$\mathbf{P}\left(SAT\right) = \mathbf{E}\left[2^{-\#\{\text{Poisson points}\}}\right] \cdot \mathbf{E}\left[2^{-\#\{\text{small unicyclic}\}}\right]$$

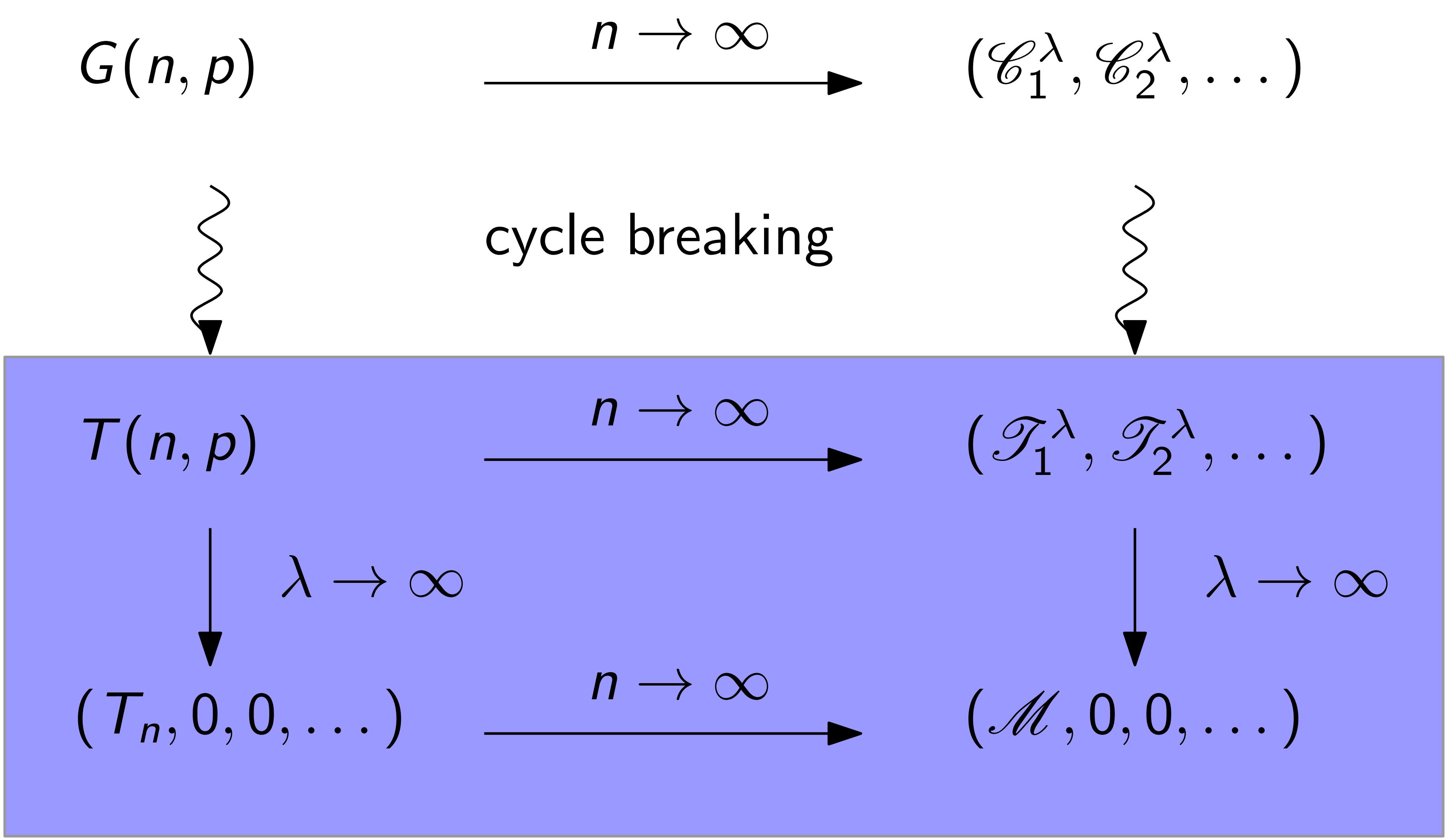
Bipartiteness / 2COL

2COL iif no cycle of odd length

$$\mathbf{P}\left(\text{length of a core edge odd}\right) \sim 1/2$$

$\Rightarrow$ Same asymptotics

Construction of the limit



Many questions

Robustness / Universality?

Random graphs with fixed degree sequence

Percolation cluster on high dimensional tori

Dynamics of the limit

Other applications?

Thank you!