# Overview of quantitative logic

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Journées Boole

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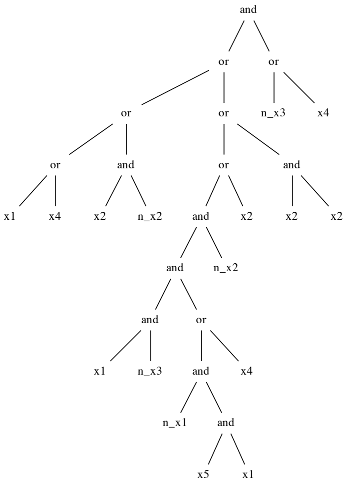
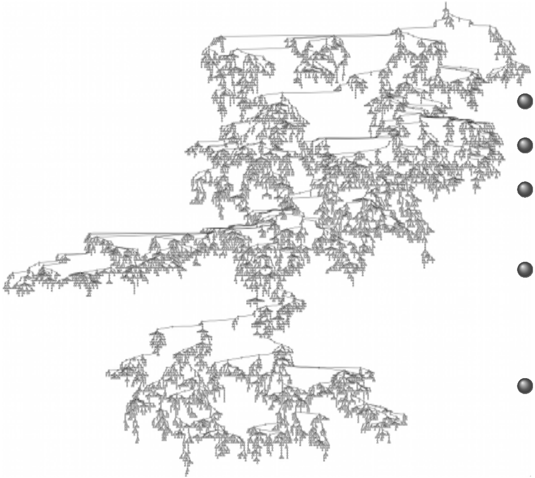
# Context

Take a Boolean formula:



# Context

Take "at random" a huge Boolean formula:



- Is it satisfiable?
- Is it a tautology?
- Which Boolean function does it represent?
- What is the complexity of this represented Boolean function?
- What is the probability of obtaining a specified Boolean function?

1. Goals
   - to build : [84+]
   - to compare : [00+]
   - to describe : [94+]

2. And what about the Boole project ? [09+]
   - Extensions of models
   - New models or results

3. Perspectives

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# Majority function: [Valiant84]

JOURNAL OF ALGORITHMS 5, 363–366 (1984)

## Short Monotone Formulae for the Majority Function

L. G. VALIANT

*Aiken Computation Laboratory, Harvard University, Cambridge, Massachusetts 02138*

Received March 18, 1983

It is shown that the monotone formula-size complexity of the monotone symmetric functions on $n$ variables can be bounded above by a function of order $O(n^{5.3})$.

# Distribution completely biased

*Ingredients:*

- a single connective
- balanced trees
- biased distribution on literals and constants

*Method:*

Probabilistic amplification
based on real analysis

- [Valiant84]: dirac for `Majority` function
  **High probability to get a formula of size $O(n^{5.3})$ that represents `Majority(n)`**
- [Boppana85]: distribution concentrated on *threshold* functions
- [GM97], [Servedio04]

# Distribution completely biased, unless . . .

## Goals: Systematization and classification

*Ingredients:*

- a set of connectives
- balanced trees
- distribution (uniform or not) on literals

*Method:*

Probabilistic amplification
based on real analysis

- [Savický90]: a single linear connective induces uniform distribution
- [BP05]: analysis of the distr. according to prop. of the single conn.
- [FGG09]: what about 2 random connectives?

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# Quantitative logics: [MTZ00]

*Under consideration for publication in Math. Struct. in Comp. Science*

## Statistical Properties of Simple Types

M. MOCZURAD,[1] J. TYSZKIEWICZ[2†] and M. ZAIONC[1‡]

We consider types and typed lambda calculus over a finite number of ground types. We are going to investigate the size of the fraction of inhabited types of the given length $n$ against the number of all types of length $n$. The plan of this paper is to find the limit of that fraction when $n \to \infty$. The answer to this question is equivalent to finding the "density" of inhabited types in the set of all types, or the so-called asymptotic probability of finding an inhabited type in the set of all types. Under the Curry-Howard isomorphism this means finding the density or asymptotic probability of provable intuitionistic propositional formulas in the set of all formulas. For types with one ground type (formulas with one propositional variable) we prove that the limit exists and is equal to $1/2 + \sqrt{5}/10$, which is approximately 72.36%. This means that a long random type (formula) has this probability to be inhabited (tautology). We also prove that for every finite number $k$ of ground-type variables, the density of inhabited types is always positive and lies between $(4k+1)/(2k+1)^2$ and $(3k+1)/(k+1)^2$. Therefore we can easily see that the density is decreasing to 0 with $k$ going to infinity. From the lower and upper bounds presented we can deduce that at least 1/3 of classical tautologies are intuitionistic.

# What do tautologies look like?

## Goal: Characterizing almost all tautologies

*Ingredients:*

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- Catalan trees

*Method:*

Exhibiting the smallest set of constraints satisfied by almost all tautologies.

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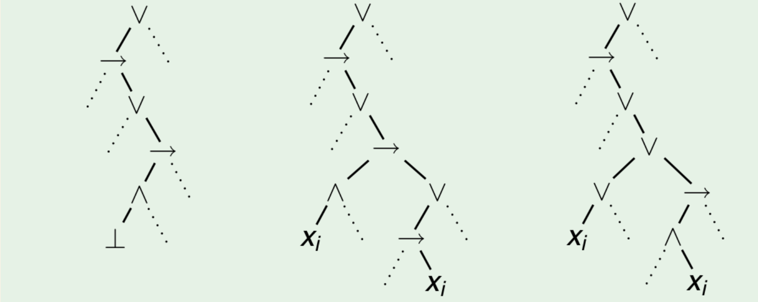
- [Zaionc05]: On the asymptotic density of tautologies in logic of implication and negation.
- [Woods06]: On the probability of absolute truth for and/or formulas.
- [FGGZ07]: Classical and intuitionistic logic are asymptotically identical.
- [KZ08]: Asymptotic densities in logic and type theory.
- [GKM08]: On the density and the structure of the Peirce-like formulae.
- [GK09]: In the full propositional logic, 5/8 of classical tautologies are intuitionistically valid.

# Full propositional logic

$\{\wedge, \vee, \rightarrow\}$ and $\{x_1, x_2, \ldots, x_k, \bot\}$



# Outline

# Large random And/Or trees: [LS97]

## Some Typical Properties of Large AND/OR Boolean Formulas

**Hanno Lefmann**

**Petr Savický[1]**

In this paper typical properties of large random Boolean AND/OR formulas are investigated. Such formulas with $n$ variables are viewed as rooted binary trees chosen from the uniform distribution of all rooted binary trees on $m$ nodes, where $n$ is fixed and $m$ tends to infinity. The leaves are labeled by literals and the inner nodes by the connectives AND/OR, both uniformly at random. In extending the investigation to infinite trees, we obtain a close relation between the formula size complexity of any given Boolean function $f$ and the probability of its occurrence under this distribution, i.e., the negative logarithm of this probability differs from the formula size complexity of $f$ only by a polynomial factor.

# What about the distribution on Boolean functions?

## Goal: Characterizing the probability of a function

*Ingredients:*

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- Catalan trees or Galton Watson trees

*Methods:*

Proving that almost all trees computing a function have a simple shape.

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**Kozik's pattern languages theory.**

- [LS97]: Some typical properties of large And/Or Boolean formulas.
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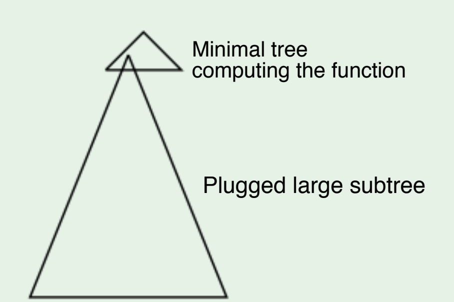
- [Gardy06]: Random Boolean expressions.
- [FGGG08]: Complexity and limiting ratio of Boolean functions over Implication.
- [Kozik08]: Subcritical pattern languages for And/Or trees.

# Main results before the Boole project started

Let $f$ be a function. In both models of Implication and And/Or formulas:

*Asymptotically, almost all trees computing $f$ have a simple form.*



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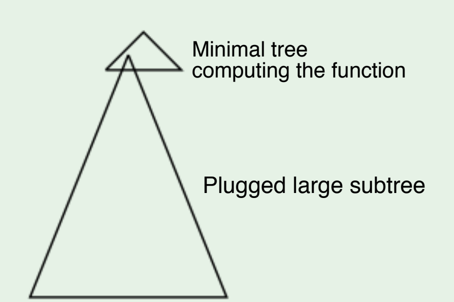


$$\lim_n \mathbb{P}_{k,n}(f) \sim_k \frac{\lambda(f)}{4^{L(f)}\, k^{L(f)+1}},$$

where $L(f)$ is the complexity of $f$ and
$\lambda(f)$ a constant depending of the minimal trees of $f$.

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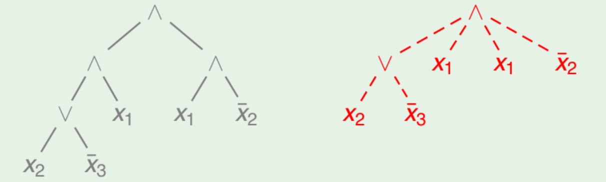
# Extension of Kozik's method

## Kozik's pattern languages

*Kozik's theory:*
Plane and binary
And/Or trees

*Extensions:*
Non-Plane or non-binary And/Or trees.
*Tree models are fundamentally different.*



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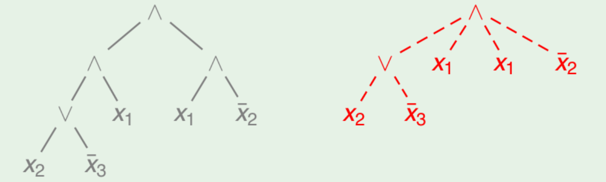


[GGKM++]: **Boolean formulas with associativity or commutativity.**

## Results

Theorems are similar as in the original model.

# Other non-plane model

## Model of Implication

*Commutation of premises:*
$A \to (B \to C) \approx B \to (A \to C)$

*Extensions:*
Non-Plane or non-binary Implication trees.

## Results

[GGKM12]: Theorems are similar a in the original model.

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# Growing trees

*Model:*
**And/Or growing trees**
inspired by
binary search trees

*Methods:*
Analytic combinatorics or
Probabilistic approaches

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## Results

[CGM11]: **Distribution concentrated on constant functions.**

# Three global results

No Shannon effect in the distributions on Boolean functions induced by random Boolean trees.

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[BM++]: Distributions induced by Boolean trees, whose saturation level tends to infinity, are concentrated on both constant functions.

[GM++]: Nb of variables linked to the size of the trees

$$\mathbb{P}_n\langle f\rangle \sim \begin{cases} \lambda_{\langle f\rangle} \cdot \left(\frac{1}{k_{n+1}}\right)^{R\langle f\rangle+1}, & \text{if, for large enough } n,\ k_n \leq n/\ln n; \\ \lambda_{\langle f\rangle} \cdot \left(\frac{\ln n}{n}\right)^{R\langle f\rangle+1} & \text{otherwise.} \end{cases}$$

where $n$ is the size, $k_n$ the nb. of variables (monotone in $n$) and $R\langle\cdot\rangle$ corresponds, in some sense, to the complexity.

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# New directions

- Meta-theorems to predict the behaviour of a model

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- Model with another notion of size/complexity
- Model based on Directed Acyclic Graphs

# Au final...

Merci pour votre attention.

J'ai eu la chance de participer à plusieurs de ces réflexions,
merci beaucoup à tous les collègues qui m'ont supporté !!

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- [GM++]: Genitrini and Mailler: Equivalence classes of random Boolean trees