Infinite Systems of Functional Equations and Gaussian Limiting Distributions

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joint work with
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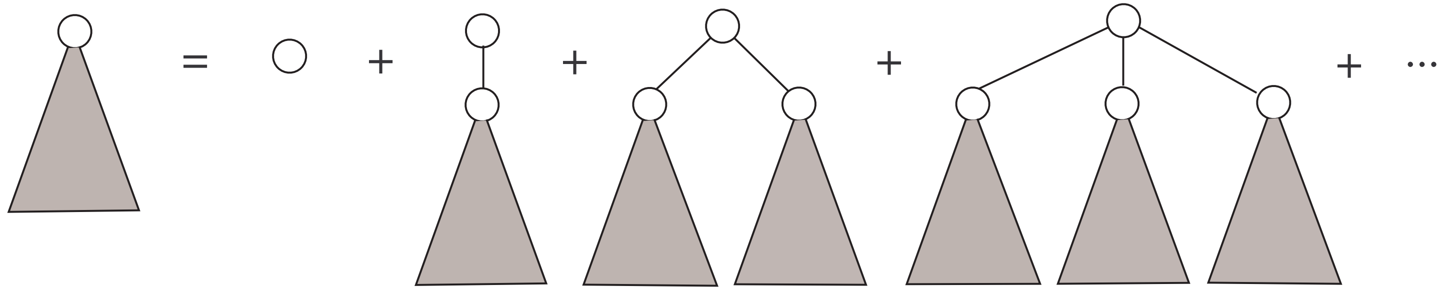
Outline

- Examples
- Preliminaries from functional analysis
- Some Probability Theory
- Implicit Functions
- Sketch of proofs
- Gaussion Limits
- Examples cont'd

Example: Rooted Plane Trees



$$y(x) = x + xy(x) + xy(x)^2 + xy(x)^3 + \cdots = \frac{x}{1 - y(x)}.$$

$\Longrightarrow$

$$y(x) = \frac{1 - \sqrt{1 - 4x}}{2}$$

and

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Example: Rooted Plane Trees

Let $\mathbf{k} = (k_0, k_1, k_2, \ldots)$ be a sequence of non-negative integers

$y_{n,\mathbf{k}}$ ... # trees of size n s.t. k_j vertices have exactly j successors

Let $\mathbf{u} = (u_0, u_1, u_2, \ldots)$ and $\mathbf{u}^{\mathbf{k}} = u_0^{k_0} u_1^{k_1} u_2^{k_2} \cdots$

Then $y(x, \mathbf{u}) = \sum_{n,\mathbf{k}} y_{n,\mathbf{k}} x^n \mathbf{u}^{\mathbf{k}}$ satisfies

$$y(x,\mathbf{u}) = \underbrace{xu_0 + xu_1 y(x,\mathbf{u}) + xu_2 y(x,\mathbf{u})^2 + xu_3 y(x,\mathbf{u})^3 + \cdots}_{F(x,y(x,\mathbf{u}),\mathbf{u})}.$$

$\|\mathbf{u}\|_\infty$ bounded $\rightsquigarrow$ analytic equation for $y(x, \mathbf{u})$.

Example: Rooted Plane Trees

$X_n^{(j)}$ ... # nodes of out-degree j in random tree of size n.

Let $\mathbf{X}_n = (X_n^{(0)}, X_n^{(1)}, X_n^{(2)}, \ldots)$; then

$$\mathbb{E}\,\mathbf{u}^{\mathbf{X}_n} = \frac{1}{y_n}[x^n]\,y(x,\mathbf{u}),$$

Let ℓ be the linear functional

$$\ell \cdot \mathbf{X}_n = \sum_{j\geq 0} \ell_j X_n^{(j)}.$$

Then

$$\mathbb{E}\,e^{it\ell\cdot\mathbf{X}_n} = \frac{1}{y_n}[x^n]\,y(x, e^{it\ell_0}, e^{it\ell_1}, \ldots).$$

Other Examples: Bipartite Planar Maps

Bouttier, Di Francesco, Guitter '04:

x counts number of edges, z counts number of vertices

u_j counts number of faces with boundary length $2j$

Let $R(x,z,\mathbf{u})$ denote the solution of

$$R = xz + x\sum_{j\geq 1} u_j \binom{2j-1}{j} R^j.$$

Then the g.f. $M(x,z,\mathbf{u})$ of bipartite maps, satisfies

$$M_x = R$$

Other Examples: Subcritical Graphs

$B(x)$ e.g.f. of 2-connected (labeled) graphs
$C(x)$ e.g.f. of connected graphs
Then

$$C'(x) = e^{B'(xC'(x))}.$$

Drmota, Fusy, Kang, Kraus, Rue '11:
number of vertices of degree j satisfy a CLT
(finite system of functional equations)

all $j \ge 1$ simultaneously:
$B_r^\bullet(x, u_1, u_1, \ldots)$ g.f. of rooted 2-connected graphs
(root vertex has degree r),
$C_j^\bullet(x, u_1, u_2, \ldots)$ g.f. of rooted connected graphs.
(root vertex has degree j), $C_0^\bullet(x, \mathbf{u}) = 1$

Other Examples: Subcritical Graphs

Then

$$C_j^\bullet(x,\mathbf{u}) = \sum_{l_1+2l_2+\cdots jl_j=j} \prod_{r=1}^{j} \frac{B_r^\bullet(x,W_1,W_2,\ldots)^{l_r}}{l_r!},$$

with

$$W_j = \sum_{i\geq 0} u_{i+j} C_i^\bullet(x,\mathbf{u})$$

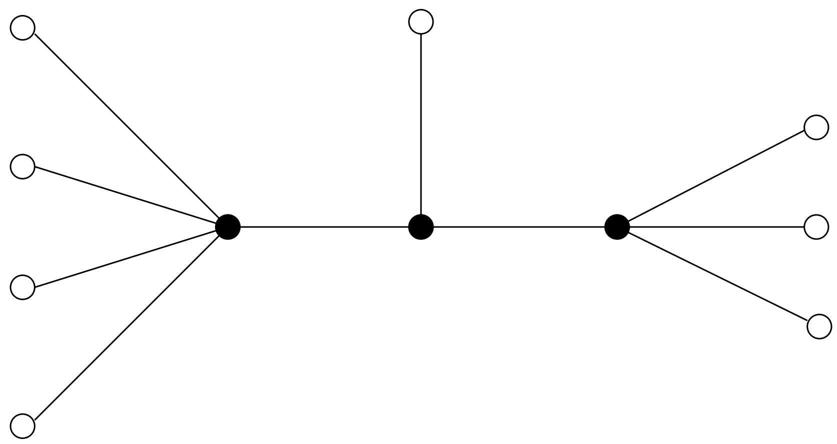
The g.f. encoding the degree distribution is

$$C^\bullet(x,\mathbf{u}) = \sum_{j\geq 0} C_j^\bullet(x,\mathbf{u}).$$

Other Examples: Patterns in Trees

Chyzak, Drmota, Klausner, Kok '08:

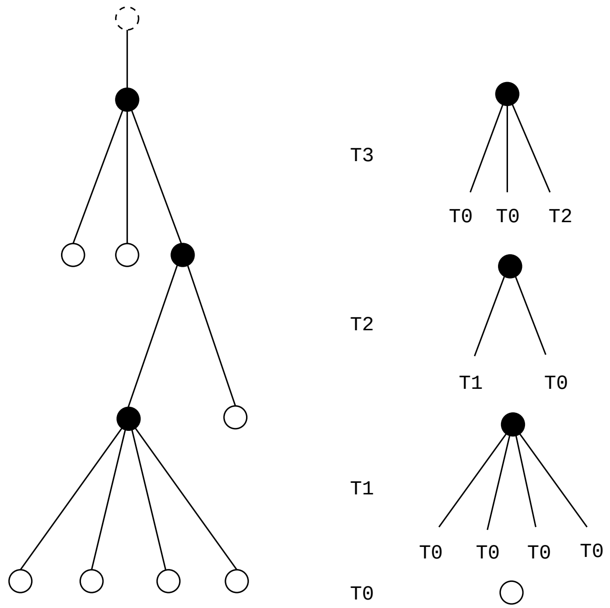
Count, e.g., occurrences of



in trees.

Other Examples: Patterns in Trees

Splitting the pattern yields partition of the class of trees



$\rightsquigarrow$ system of functional equations

infinitely many patterns simultaneously
$\rightsquigarrow$ infinite system of functional equations

Preliminaries

We consider linear operators $T : B_1 \to B_2$, where B_1 and B_2 are Banach spaces

U is the open unit ball in B_1

$T : B_1 \to B_2$ compact $\Leftrightarrow$ closure of $T(U)$ is compact in B_2
$\Leftrightarrow$ every bounded $(x_n)_{n\in\mathbb{N}}$ has subsequence $(x_{n_i})_{i\in\mathbb{N}}$ s.t. $(Tx_{n_i})_{i\in\mathbb{N}}$ converges in B_2

$A : B \to B$ bounded operator;

$$r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$$

denotes its spectral radius, $\sigma(A)$ its spectrum.

Preliminaries

Let $1 \leq p < \infty$: ℓ^p ... set of $\mathbb{C}$-valued sequences $(t_n)_{n\in\mathbb{N}}$
s.t. $\|(t_n)\|_p^p := \sum_{n=1}^{\infty} |t_n|^p < \infty$.

In ℓ^p:
bounded linear operator A $\Leftrightarrow$ infinite dimensional matrix $(a_{ij})_{1\leq i,j<\infty}$
(Not true for ℓ^∞!)

A **positive**, if $a_{ij} \geq 0$

A **irreducible**, if A is positive and for all (i,j) there is an $n > 0$ s.t. $(A^n)_{ij} > 0$

Dual space of ℓ^p is ℓ^q with $\frac{1}{p} + \frac{1}{q} = 1$.

Some Probability Theory

$\mathbf{X}$ ... random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to a Banach space B
P ... law of $\mathbf{X}$
ℓ^* continuous functional
$\Longrightarrow \ell^*(\mathbf{X})$ determine the distribution of $\mathbf{X}$

$\mathbf{X}_n \xrightarrow{w} \mathbf{X}$ if $P_n \xrightarrow{w} P$, i.e., if

$$\int_B f \, \mathrm{d}P_n \to \int_B f \, \mathrm{d}P$$

for every bounded continuous real function f.

A set Π of probability measures is **tight** if for each $\varepsilon > 0$ there is a compact set K s.t. $P(K) > 1 - \varepsilon$ for every $P \in \Pi$.

Some Probability Theory

Theorem 1

$$\mathbf{X}_n \xrightarrow{w} \mathbf{X}$$

if $\{P_n : n \in \mathbb{N}\}$ *is tight and*

$$\mathbb{E}\left[e^{it\ell^*(\mathbf{X}_n)}\right] \to \mathbb{E}\left[e^{it\ell^*(\mathbf{X})}\right]$$

for all $\ell^* \in B^*$,

$\mathbf{X}$ **Gaussian** if $\ell^*(\mathbf{X})$ is Gaussian variable for all $\ell^* \in B^*$.

Then $\mathbb{E}\mathbf{X} = \mathbf{y} \in B$ s.t.

$$\ell^*(\mathbf{y}) = \mathbb{E}(\ell^*(\mathbf{X})).$$

for all $\ell^* \in B^*$.

Implicit Functions

Set $\mathbf{u} = e^{\mathbf{v}} := (e^{v_1}, e^{v_2}, \dots)$ and fix p, r with $1 \leq p < \infty$, $1 \leq r \leq \infty$.

Theorem 2 *Let* $\mathbf{F} : \mathbb{C} \times \ell^p \times \ell^r \to \ell^p$, $(x, \mathbf{y}, \mathbf{v}) \mapsto \mathbf{F}(x, \mathbf{y}, \mathbf{v})$ *s.t.:*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}, \mathbf{0}) = A(x, \mathbf{y}) + \alpha(x, \mathbf{y})\, \mathbf{I}_{\mathsf{p}},$$

$\alpha(x, \mathbf{y})$ *is analytic and positive,* A *irred.,* A^n *compact for some* n*;*

Let $\mathbf{y} = \mathbf{y}(x, \mathbf{v})$ *be the unique solution of* $\begin{cases} \mathbf{y} = \mathbf{F}(x, \mathbf{y}, \mathbf{v}), \\ \mathbf{y}(0, \mathbf{v}) = \mathbf{0}. \end{cases}$
For $\mathbf{y}(x, \mathbf{0})$ *assume rad. of conv.* $x_0 > 0$ *and that* $\mathbf{y}_0 := \mathbf{y}(x_0, \mathbf{0})$ *exists. Then*

$$\mathbf{y}(x, \mathbf{v}) = \mathbf{g}(x, \mathbf{v}) - \mathbf{h}(x, \mathbf{v}) \sqrt{1 - \frac{x}{x_0(\mathbf{v})}}$$

Under aperiodicity conditions we have also Δ*-analyticity.*

Sketch of the Proof

First, let $\mathbf{v} = \mathbf{0}$.
Implicit fctn theorem + Banach fixed-point theorem:

$$\mathbf{y}^{(n+1)}(x, \mathbf{0}) = \mathbf{F}(x, \mathbf{y}^{(n)}(x, \mathbf{0}), \mathbf{0})$$

converges uniformly to the unique solution $\mathbf{y}(x, \mathbf{0})$.
$\mathbf{F}$ positive $\Longrightarrow$ $\mathbf{y}(x, \mathbf{0})$ positive.

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$\mathbf{F}$ positive $\Longrightarrow$ $\mathbf{y}(x, \mathbf{0})$ positive.

Next:

$$\mathbf{y}_0 = \mathbf{F}(x_0, \mathbf{y}_0, \mathbf{0}), \qquad r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x_0, \mathbf{y}_0, \mathbf{0})\right) = 1.$$

We use that $x \mapsto r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right)$ is strictly increasing and that $r\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right) < 1$ for $x < x_0$

follows from $\left(I - \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})\right) \frac{\partial \mathbf{y}}{\partial x}(x, \mathbf{0}) = \frac{\partial \mathbf{F}}{\partial x}(x, \mathbf{y}(x, \mathbf{0}), \mathbf{0})$

Sketch of the Proof

Third step: Let $\mathbf{y} = (y_1, \overline{\mathbf{y}})$ and $\mathbf{F} = (F_1, \overline{\mathbf{F}})^t$.

$$y_1 = F_1(x, y_1, \overline{\mathbf{y}}, \mathbf{0}),$$
$$\overline{\mathbf{y}} = \overline{\mathbf{F}}(x, y_1, \overline{\mathbf{y}}, \mathbf{0}).$$

Jacobian operators:

$$A = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x_0, \mathbf{y}_0, \mathbf{0}) \qquad B = \frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x_0, y_{01}, \overline{\mathbf{y}_0}, \mathbf{0}).$$

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Note:

$$r(B) < r(A) = 1.$$

Insert solution of second equation into first one:

$$y_1 = F_1(x, y_1, \overline{\mathbf{y}}(x, y_1, \mathbf{0}), \mathbf{0}) =: G(x, y_1).$$

Equation for $y_1 = y_1(x, \mathbf{0})$.

Sketch of the Proof

4th step: Find singularity of solution y_1.

$$y_1 = G(x, y_1),$$
$$1 = G_{y_1}(x, y_1),$$

with $G_x(x_0, y_{01}) \neq 0$ and $G_{y_1 y_1}(x_0, y_{01}) \neq 0$.

$\Longrightarrow$

$$y_1(x, \mathbf{0}) = g_1(x, \mathbf{0}) - h_1(x, \mathbf{0})\sqrt{1 - \frac{x}{x_0}}$$

Insert this solution into $\overline{y}$

$\Longrightarrow$

$$\overline{\mathbf{y}}(x, y_1(x, \mathbf{0}), \mathbf{0}) = \overline{\mathbf{g}}(x, \mathbf{0}) - \overline{\mathbf{h}}(x, \mathbf{0})\sqrt{1 - \frac{x}{x_0}}.$$

Sketch of the Proof

5th step: analyticity in $\mathbf{v}$. From

$$r\left(\frac{\partial \overline{\mathbf{F}}}{\partial \overline{\mathbf{y}}}(x_0, (\mathbf{y_0})_1, \overline{\mathbf{y_0}}, \mathbf{0})\right) < 1.$$

we get unique solution $\overline{\mathbf{y}}(x, y_1, \mathbf{v})$ of the functional equation

$$\overline{\mathbf{y}} = \overline{\mathbf{F}}(x, y_1, \overline{\mathbf{y}}, \mathbf{v})$$

near $(x_0, (\mathbf{y_0})_1, \mathbf{0})$.

$\Longrightarrow$

$$\begin{aligned} y_1 &= G(x, y_1, \mathbf{v}), \\ 1 &= G_{y_1}(x, y_1, \mathbf{v}), \end{aligned}$$

has positive solution $(x_0, (\mathbf{y_0})_1, \mathbf{0})$. Furthermore,

$$\det\begin{pmatrix} -G_x & 1 - G_{y_1} \\ -G_{y_1,x} & -G_{y_1,y_1} \end{pmatrix} = G_x \cdot G_{y_1 y_1} \neq 0,$$

$\Longrightarrow$ there exist $x_0(\mathbf{v})$ and $y_1(\mathbf{v})$ near $\mathbf{0}$.

Gaussian Limits

$$y(x,\mathbf{v}) = G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \sum_{n=0}^{\infty} c_n(\mathbf{v})x^n,$$

$\mathbf{X}_n$ corresponding ℓ^p-valued random variable then

$$\mathbb{E}\left[e^{it\boldsymbol{\ell}\cdot\mathbf{X}_n}\right] = \frac{c_n(it\boldsymbol{\ell})}{c_n(\mathbf{0})}$$

for all $\boldsymbol{\ell} \in \ell^q$.

In applications:

$$G(x,\mathbf{y}(x,\mathbf{v}),\mathbf{v}) = \sum_{n=0}^{\infty} \sum_{\mathbf{m}\in\ell^p} c_{n,\mathbf{m}} e^{\mathbf{m}\cdot\mathbf{v}} x^n = \sum_{n=0}^{\infty} \sum_{\mathbf{m}\in\ell^p} c_{n,\mathbf{m}} \mathbf{u}^{\mathbf{m}} x^n,$$

where $c_{n,\mathbf{m}}$ denotes # objects of size n and characteristic $\mathbf{m}$.

Gaussian Limits

Corollary $\mathbf{y} = \mathbf{y}(x, \mathbf{v})$ *as before,* $G : (\mathbb{C}, \ell^p, \ell^r) \to \mathbb{C}$ *analytic at* $(x_0(\mathbf{0}), \mathbf{y}(x_0(\mathbf{0}), \mathbf{0}), \mathbf{0})$*,*

$$\frac{\partial G}{\partial \mathbf{y}}(x_0(\mathbf{0}), \mathbf{y}(x_0(\mathbf{0}), \mathbf{0}), \mathbf{0}) \neq \mathbf{0}.$$

Then

$$G(x, \mathbf{y}(x, \mathbf{v}), \mathbf{v}) = \bar{g}(x, \mathbf{v}) - \bar{h}(x, \mathbf{v})\sqrt{1 - \frac{x}{x_0(\mathbf{v})}}$$

Moreover,

$$[x^n]G(x, \mathbf{y}(x, \mathbf{v}), \mathbf{v}) \sim \frac{\bar{h}(x_0(\mathbf{v}), \mathbf{v})}{2\sqrt{\pi}} x_0(\mathbf{v})^{-n} n^{-3/2}$$

uniformly for $\mathbf{v}$ *in a neighborhood of* $\mathbf{0}$*.*

Gaussian Limits

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Proof:

$$G(x, \mathbf{y}, \mathbf{v}) = G(x, \mathbf{g}(x, \mathbf{v}), \mathbf{v}) + \sum_{n \geq 1} \frac{A_n(x, \mathbf{v})}{n!} \Big((\mathbf{y} - \mathbf{g}(x, \mathbf{v}))^n \Big).$$

Gaussian Limits

Theorem 3 $1 \le p < \infty$, $\mathbf{X}_n$ *is a sequence of* ℓ^p*-valued random variables as above,* $\boldsymbol{\ell} \in \ell^q$*.*
Then $\boldsymbol{\ell} \cdot \mathbb{E}\mathbf{X}_n = \mu_{\boldsymbol{\ell}} n + O(1)$ *with* $\mu_{\boldsymbol{\ell}} = -\frac{\partial x_0}{\partial \mathbf{v}}(\mathbf{0}) \cdot \boldsymbol{\ell}/x_0$ *and*

$$\boldsymbol{\ell} \cdot \left(\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \right) \xrightarrow{w} \mathcal{N}(\mathbf{0}, \boldsymbol{\ell}^T B \boldsymbol{\ell})$$

where $B \in L(\ell^q, \ell^p)$ *is given by the matrix*

$$\frac{1}{x_0^2} \left(\frac{\partial x_0}{\partial v_i}(\mathbf{0}) \cdot \frac{\partial x_0}{\partial v_j}(\mathbf{0})^T \right)_{1 \le i,j < \infty} - \frac{1}{x_0} \left(\frac{\partial^2 x_0}{\partial v_i v_j}(\mathbf{0}) \right)_{1 \le i,j < \infty} .$$

Corollary *If in addition the set of laws of* $(\mathbf{X}_n - \mathbb{E}\mathbf{X}_n)/\sqrt{n}$ *is tight then there exists a centered Gaussian random variable* $\mathbf{X}$ *such that*

$$\frac{\mathbf{X}_n - \mathbb{E}\mathbf{X}_n}{\sqrt{n}} \xrightarrow{w} \mathbf{X},$$

Tightness

Theorem 4 *Single functional equation* $y = F(x, y, \mathbf{v})$ *where* $F : B \times U \times V \to \mathbb{C}$ *is a positive and analytic and* $B \times U \times V \subseteq \mathbb{C}^2 \times \ell^2$ *such that there exist positive real* $(x_0, y_0) \in B \times U$ *with* $y_0 = F(x_0, y_0, \mathbf{0})$ *and* $1 = F_y(x_0, y_0, \mathbf{0})$ *such that* $F_x(x_0, y_0, \mathbf{0}) \neq 0$ *and* $F_{yy}(x_0, y_0, \mathbf{0}) \neq 0$.

Furthermore assume that the corresponding r.v.'s $X_n^{(j)}$ *fulfil* $X_n^{(j)} = 0$ *for* $j > cn$ *and that:*

$$\sum_{j\ge 0} F_{v_j} < \infty, \quad \sum_{j\ge 0} F_{yv_j}^2 < \infty, \quad \sum_{j\ge 0} F_{v_jv_j} < \infty,$$

$$F_{xv_j} = o(1), \quad F_{xv_jv_j} = o(1), \quad F_{yyv_j} = o(1), \quad F_{yyv_jv_j} = o(1),$$

$$F_{xxv_j} = O(1), \quad F_{xyv_j} = O(1), \quad F_{xyyv_j} = O(1), \quad F_{yyyv_j} = O(1),$$

all derivatives evaluated at $(x_0, y_0, \mathbf{0})$. $\Longrightarrow$ *tightness.*

Sketch of Proof

By a theorem of Grenander (1963) tightness follows from

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbb{E}\left[\sum_{j>N} \frac{(X_n^{(j)} - \mathbb{E}X_n^{(j)})^2}{n}\right] = 0.$$

The assumption $X_n^{(j)} = 0$ if $j > cn$ implies that this reduces to

$$\lim_{N\to\infty} \sup_{n\geq 1} \sum_{N<j\leq cn} \sigma_{n,j}^2 = 0,$$

where $\sigma_{n,j}^2 = \mathbf{Var}(X_n^{(j)} - \mathbb{E}X_n^{(j)})/\sqrt{n}$.

Tightness follows from $\sigma_{n,j}^2 = \sigma_j^2 + \tau_j/n + O(n^{-2})$ if

$$\sum_{j\geq 0} \sigma_j^2 < \infty, \quad \text{and} \quad \tau_j = o(1) \quad (j \to \infty),$$

uniformly in j.

Examples cont'd: Subcritical Graphs

Recall

$$C_j^\bullet(x,\mathbf{u}) = \sum_{l_1+2l_2+\cdots jl_j=j} \prod_{r=1}^{j} \frac{B_r^\bullet(x,W_1,W_2,\ldots)^{l_r}}{l_r!},$$

with

$$W_j = \sum_{i\geq 0} u_{i+j} C_i^\bullet(x,\mathbf{u})$$

The g.f. of interest is: $C^\bullet(x,\mathbf{u}) = \sum_{j\geq 0} C_j^\bullet(x,\mathbf{u})$.

Examples cont'd: Subcritical Graphs

Recall

$$C_j^\bullet(x,\mathbf{u}) = \sum_{l_1+2l_2+\cdots jl_j=j} \prod_{r=1}^{j} \frac{B_r^\bullet(x,W_1,W_2,\ldots)^{l_r}}{l_r!},$$

with

$$W_j = \sum_{i\geq 0} u_{i+j} C_i^\bullet(x,\mathbf{u})$$

The g.f. of interest is: $C^\bullet(x,\mathbf{u}) = \sum_{j\geq 0} C_j^\bullet(x,\mathbf{u})$.

Only difficulty:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x,\mathbf{y},\mathbf{0}) = A(x,\mathbf{y}) + \alpha(x,\mathbf{y})\,\mathrm{I_p},$$

α analytic and positive, A irreducible, A^n compact for some n;

Examples cont'd: Subcritical Graphs

Lemma 1 *Let* $H(x, y, w)$ *be a positive function, suppose that* $y(x)$ *has finite radius of convergence* x_0 *(so that* $H(x, y, 1)$ *is analytic at* (x_0, y_0)*) and satisfies* $y(x) = H(x, y(x), 1)$*.*

Consider the system

$$y_j(x, \mathbf{u}) = F_j(x, \mathbf{y}(x, \mathbf{u}), \mathbf{u})$$

and assume

$$F_i(x, \mathbf{y}, \mathbf{1}) = [w^i]H\left(x, \sum_j y_j, w\right).$$

Then we have $y(x) = \sum_i y_i(x, \mathbf{1})$ *and the operator* $A = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, y, \mathbf{1})$ *is compact.*

Examples cont'd: Subcritical Graphs

Lemma 2 *Let $H(x, y, w)$ be a positive function, suppose that $y(x)$ has finite radius of convergence x_0 (so that $H(x, y, 1)$ is analytic at (x_0, y_0)) and satisfies $y(x) = H(x, y(x), 1)$.*

Consider the system

$$y_j(x, \mathbf{u}) = F_j(x, \mathbf{y}(x, \mathbf{u}), \mathbf{u})$$

and assume

$$F_i(x, \mathbf{y}, \mathbf{1}) = [w^i]H\left(x, \sum_j y_j, w\right).$$

Then we have $y(x) = \sum_i y_i(x, \mathbf{1})$ and the operator $A = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(x, y, \mathbf{1})$ is compact.

Proof. $y(x) = \sum_i y_i(x, \mathbf{1})$ is obvious.

$\frac{\partial F_i}{\partial y_j} = a_i(x, \sum_\ell y_\ell) \cdot y_j$, thus A has rank 1 and is compact.

Examples cont'd: Subcritical Graphs

In our example: $C'(x) = C^\bullet(x, \mathbf{1})$, $B'(x) = B^\bullet(x, \mathbf{1})$

Set $y_j(x, \mathbf{u}) = C_j^\bullet(x, \mathbf{u})$; then $y = \sum_j y_j$.

Choose

$$H(x, y, w) = \sum_j y_j w^j = \mathrm{exp}\left(\sum_k B_k^\bullet(xy, \mathbf{1}) w^k\right)$$

$$H(x, C'(x), w) = \mathrm{exp}\left(\sum_k B_k^\bullet(xC'(x), \mathbf{1}) w^k\right)$$

Then

$$H(x, y, 1) = e^{B'(xy)}$$

which corresponds to $C'(x) = e^{B'(xC'(x))}$.

Examples cont'd: Random Walks on Groups

$\Gamma = \Gamma_1 * \Gamma_2 * \cdots := (\bigcup_i(\Gamma_i \setminus \{1\}))^*$ free product of groups

random walk on Γ: $S_n = \xi_1\xi_2\cdots\xi_n$
where ξ_i are i.i.d with common distribution

$$P\{\xi_1 = \alpha\} = p_i q_\alpha \text{ if } \alpha \in \Gamma_i \setminus \{1\} \qquad \text{and } P\{\xi_1 = \varepsilon\} = p_0$$

where $p_i > 0$, $i \in \mathbb{N}$, and $q_\alpha > 0$ and $\{n : P\{S_n = \varepsilon\} > 0\} \not\subseteq U < \Gamma$

Theorem 5 (Lalley '02) *There exist $C > 0$ and $1 < R < \infty$ such that $P\{S_n = \varepsilon\} \sim CR^{-n}n^{-3/2}$.*

Proof based on the generating functions for first hitting times: $\tau_\alpha = \min\{n \geq 0 : S_n = \alpha\}$ $(\alpha \in \Gamma_i)$ and

$$y_{i;\alpha(x)} = \sum_{n\geq 1} P\{\tau_\alpha = n\}x^n,$$

Examples cont'd: Random Walks on Groups

Lalley showed

$$y_{i;\alpha}(x) = x\Bigg(p_i q_\alpha + p_0 y_{i;\alpha}(x) + \sum_{\gamma\in\Gamma_i\setminus\{\alpha\}} p_i q_\gamma y_{i;\gamma^{-1}\alpha}(x)$$

$$+ \sum_{j\neq i}\sum_{\gamma\in\Gamma_i} p_j q_\gamma y_{j;\gamma^{-1}}(x) y_{i;\alpha}(x)\Bigg).$$

Furthermore, he showed: $\mathbf{y}(x) \in \ell^1$ and

$$\mathcal{J}(x) = \mathcal{K}(x) + \mathcal{L}(x) + \frac{G(x)-1}{xG(x)}\mathbf{I}_1,$$

where

$$G(x) = \sum_{n\geq 1} P\{S_n = \varepsilon\}x^n = \left(1 - p_0 x - \sum_{\alpha\neq\varepsilon} p_\alpha x y_{\alpha^{-1}}(x)\right)^{-1},$$

MERCI!

THANK YOU!